188 (binary search again) The problem is binary search (Exercise 187), but each iteration tests to see if the item in the middle of the remaining segment is the item we seek.

(a) Write the program, with specifications and proofs.

(b) Find the execution time according to the recursive measure.

(c) Find the execution time according to a measure that charges time 1 for each test.

(d) Compare the execution time to binary search without the test for equality each iteration.

After trying the question, scroll down to the solution.

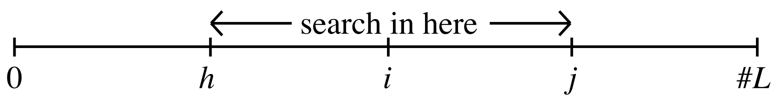
(a) Write the program, with specifications and proofs.

§ Let the list be L and the value we are looking for be x (these are not state variables). Our program will assign natural variable i the index of an occurrence of x in L if x is there. Let's indicate whether x is present in L by assigning binary variable p the value $\top$ if it is and $\bot$ if not. Ignoring time for the moment, the problem is

$$x\text{: } L\,(0,..\#L) \;=\; p' \;\Longrightarrow\; L\,i' = x$$

As the search progresses, we narrow the segment of the list that we need to search. Let us introduce natural variables h and j, and let specification R describe the search within the segment $h,..j$.

$$R \;=\; (x\text{: } L\,(h,..j) \;=\; p' \;\Longrightarrow\; L\,i' = x)$$



We can now solve the problem.

$$(x\text{: } L(0,..\#L) \;=\; p' \;\Longrightarrow\; L\,i' = x) \;\Longleftarrow\; h{:=}\,0.\;\; j{:=}\,\#L.\;\; h{<}j \Rightarrow R$$

$$\begin{aligned}h{<}j \Rightarrow R \;\Longleftarrow\; & i{:=}\,div\,(h{+}j)\,2.\\ & \textbf{if } L\,i = x \textbf{ then } p{:=}\,\top\\ & \textbf{else if } L\,i < x \textbf{ then } h{:=}\,i{+}1 \textbf{ else } j{:=}\,i \textbf{ fi}.\\ & \quad h{\le}j \Rightarrow R \textbf{ fi}\end{aligned}$$

$$h{\le}j \Rightarrow R \;\Longleftarrow\; \textbf{if } h{=}j \textbf{ then } p{:=}\,\bot \textbf{ else } h{<}j \Rightarrow R \textbf{ fi}$$

Before proving these three refinements, here are two little theorems that will be useful; let's call the first one Lemma A:

$$h{<}j \;=\; div\,(h{+}j)\,2 + 1 \le j$$

Here is its proof:

$$\begin{aligned}&\textbf{if } even\,(h{+}j)\\ &\textbf{then}\;\; div\,(h{+}j)\,2 + 1 \le j \;=\; (h{+}j)/2 + 1 \le j \;=\; h{+}2 \le j \;=\; h{<}j \qquad \text{note below}\\ &\textbf{else}\;\; div\,(h{+}j)\,2 + 1 \le j \;=\; (h{+}j{-}1)/2 + 1 \le j \;=\; h{+}1 \le j \;=\; h{<}j \textbf{ fi}\end{aligned}$$

The “note below” refers to the fact that when $h{+}j$ is even, $h{+}2 \le j \;=\; h{<}j$.

And of course we'll call the other little theorem Lemma B:

$$h{<}j \;\Longrightarrow\; h \le div\,(h{+}j)\,2$$

Here is its proof:

$$\begin{aligned}&\textbf{if } even\,(h{+}j)\\ &\textbf{then}\;\; h \le div\,(h{+}j)\,2 \;=\; h \le (h{+}j)/2 \;=\; h{\le}j \;\Longleftarrow\; h{<}j\\ &\textbf{else}\;\; h \le div\,(h{+}j)\,2 \;=\; h \le (h{+}j{-}1)/2 \;=\; h{+}1 \le j \;=\; h{<}j \textbf{ fi}\end{aligned}$$

For the proof of the first refinement we start with the right side:

$$\begin{aligned}& h{:=}\,0.\;\; j{:=}\,\#L.\;\; h{<}j \Rightarrow R && \text{replace } R \text{ and then use Substitution Law twice}\\ =\;& 0{<}\#L \Rightarrow (x\text{: } L(0,..\#L) \;=\; p' \;\Longrightarrow\; L\,i' = x) && \text{we are given that } L \text{ is nonempty}\\ =\;& (x\text{: } L(0,..\#L) \;=\; p' \;\Longrightarrow\; L\,i' = x)\end{aligned}$$

The proof of the middle refinement, starting with the right side:

$$\begin{aligned}& i{:=}\,div\,(h{+}j)\,2.\\ & \textbf{if } L\,i = x \textbf{ then } p{:=}\,\top\\ & \textbf{else if } L\,i < x \textbf{ then } h{:=}\,i{+}1 \textbf{ else } j{:=}\,i \textbf{ fi}.\\ & \quad h{\le}j \Rightarrow R \textbf{ fi} && \text{replace } p \text{ assignment and } R\end{aligned}$$

$=\quad i{:=}\ div\ (h{+}j)\ 2.$
$\quad\textbf{if}\ L\,i = x\ \textbf{then}\ p'{=}\top \land h'{=}h \land i'{=}i \land j'{=}j$
$\quad\textbf{else if}\ L\,i < x\ \textbf{then}\ h{:=}\ i{+}1\ \textbf{else}\ j{:=}\ i\ \textbf{fi}.$
$\qquad h{\le}j \Rightarrow (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)\ \textbf{fi}$ distribute final line back

$=\quad i{:=}\ div\ (h{+}j)\ 2.$
$\quad\textbf{if}\ L\,i = x\ \textbf{then}\ p' \land h'{=}h \land i'{=}i \land j'{=}j$
$\quad\textbf{else if}\ L\,i < x\ \textbf{then}\ h{:=}\ i{+}1.\ \ h{\le}j \Rightarrow (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$
$\qquad\textbf{else}\ j{:=}\ i.\ \ h{\le}j \Rightarrow (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)\ \textbf{fi fi}$
substitution law twice on final two lines

$=\quad i{:=}\ div\ (h{+}j)\ 2.$
$\quad\textbf{if}\ L\,i = x\ \textbf{then}\ p' \land h'{=}h \land i'{=}i \land j'{=}j$
$\quad\textbf{else if}\ L\,i < x\ \textbf{then}\ i{+}1{\le}j \Rightarrow (x{:}\ L(i{+}1,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$
$\qquad\textbf{else}\ h{\le}i \Rightarrow (x{:}\ L(h,..i)\ =\ p'\ \Longrightarrow\ L\,i' = x)\ \textbf{fi fi}$ substitution law again

$=\quad\textbf{if}\ L(div\ (h{+}j)\ 2){=}x\ \textbf{then}\ p' \land h'{=}h \land i'{=}div\ (h{+}j)\ 2 \land j'{=}j$
$\quad\textbf{else if}\ L(div\ (h{+}j)\ 2){<}x$
$\qquad\textbf{then}\ div\ (h{+}j)\ 2 + 1{\le}j \Rightarrow (x{:}\ L(div\ (h{+}j)\ 2 + 1,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$
$\qquad\textbf{else}\ h \le div\ (h{+}j)\ 2 \Rightarrow (x{:}\ L(h,..div\ (h{+}j)\ 2)\ =\ p'\ \Longrightarrow\ L\,i' = x)\ \textbf{fi fi}$
case analysis

$=\qquad L(div\ (h{+}j)\ 2){=}x \land p' \land h'{=}h \land i'{=}div\ (h{+}j)\ 2 \land j'{=}j$
$\quad\lor\qquad L(div\ (h{+}j)\ 2){<}x$
$\qquad\land\quad (div\ (h{+}j)\ 2 + 1{\le}j \Rightarrow (x{:}\ L(div\ (h{+}j)\ 2 + 1,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x))$
$\quad\lor\qquad L(div\ (h{+}j)\ 2){>}x$
$\qquad\land\quad (h \le div\ (h{+}j)\ 2 \Rightarrow (x{:}\ L(h,..div\ (h{+}j)\ 2)\ =\ p'\ \Longrightarrow\ L\,i' = x))$

We must prove that this disjunction implies the left side of the refinement

$h{<}j \Rightarrow R$

By an antidistributive law, that is equivalent to proving that each of the three disjuncts implies the left side of the refinement. Since the left side of the refinement is an implication, by portation, we prove that its antecedent $h{<}j$, and a disjunct, implies its consequent R , which is $x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x$. So, $h{<}j$ and first disjunct:

$\quad h{<}j \land L(div\ (h{+}j)\ 2){=}x \land p' \land h'{=}h \land i' = div\ (h{+}j)\ 2 \land j'{=}j$ context
$=\quad h'{=}h{<}j{=}j' \land L\,i' = x \land p' \land\ i' = div\ (h{+}j)\ 2$ $x = L\,i'$ and $i'{=}div\ (h{+}j)\ 2{:}\ h,..j$
$\Longrightarrow\quad (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$

Middle disjunct:

$\qquad h{<}j \land L(div\ (h{+}j)\ 2){<}x$
$\quad\land\quad (div\ (h{+}j)\ 2 + 1{\le}j \Rightarrow (x{:}\ L(div\ (h{+}j)\ 2 + 1,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x))$
Use Lemma A for discharge
$=\quad h{<}j \land L(div\ (h{+}j)\ 2){<}x\ \land\ (x{:}\ L(div\ (h{+}j)\ 2 + 1,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$
If $h{<}j$ and $L(div\ (h{+}j)\ 2){<}x$ and L is sorted,
then $x{:}\ L(div\ (h{+}j)\ 2 + 1,..j)\ =\ x{:}\ L(h,..j)$
$=\quad h{<}j \land L(div\ (h{+}j)\ 2){<}x\ \land\ (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$ specialize
$\Longrightarrow\quad (x{:}\ L(h,..j)\ =\ p'\ \Longrightarrow\ L\,i' = x)$

Last disjunct:

$\qquad h{<}j \land L(div\ (h{+}j)\ 2){>}x$

$\wedge \quad (h \le div\ (h+j)\ 2 \Rightarrow (x\text{: } L(h,..div\ (h+j)\ 2) = p' \Longrightarrow L\ i' = x))$

Use Lemma B for discharge

$= \quad h<j \wedge L(div\ (h+j)\ 2)>x \ \wedge\ (x\text{: } L(h,..div\ (h+j)\ 2) = p' \Longrightarrow L\ i' = x)$

If $h<j$ and $L(div\ (h+j)\ 2)>x$ and L is sorted, then $x\text{: } L(h,..div\ (h+j)\ 2) = x\text{: } L(h,..j)$

$= \quad h<j \wedge L(div\ (h+j)\ 2)>x \ \wedge\ (x\text{: } L(h,..j) = p' \Longrightarrow L\ i' = x)$ specialize

$\Longrightarrow (x\text{: } L(h,..j) = p' \Longrightarrow L\ i' = x)$

That was a lot of work for a small program, but proving these refinements caused me to find and fix two bugs.

(b) Find the execution time according to the recursive measure.

§ For recursive execution time, put $t:= t+1$ before the final, recursive call. We prove

$T \Longleftarrow h:= 0.\ j:= \#L.\ U$

$U \Longleftarrow i:= div\ (h+j)\ 2.$
if $L\ i = x$ **then** $p:= \top$
else if $L\ i < x$ **then** $h:= i+1$ **else** $j:= i$ **fi**.
V **fi**

$V \Longleftarrow$ **if** $h=j$ **then** $p:= \bot$ **else** $t:= t+1.\ U$ **fi**

I know it's log time, but to get a good upper bound I tried a few list sizes by hand, with the value sought x less than all list values. So T , U , and V are defined as

$T = t' \le t + floor\ (log\ (\#L))$
$U = h<j \Rightarrow t' \le t + floor\ (log\ (j–h))$
$V =$ **if** $h=j$ **then** $t'=t$ **else** $t:= t+1.\ U$ **fi**
$=$ **if** $h=j$ **then** $t'=t$ **else** $t:= t+1.\ h<j \Rightarrow t' \le t + floor\ (log\ (j–h))$ **fi**
$=$ **if** $h=j$ **then** $t'=t$ **else** $h<j \Rightarrow t' \le t + 1 + floor\ (log\ (j–h))$ **fi**

where $floor$ is the function that rounds down.

Before proving the refinements, here's another useful theorem, which I'll call Lemma C.

$1 + floor\ (log\ (j – div\ (h+j)\ 2 – 1)) \ \le\ floor\ (log\ (j–h))$.

Here's the proof.

if $even\ (h+j)$
then $1 + floor\ (log\ (j – div\ (h+j)\ 2 – 1))$
$= floor\ (1 + log\ (j – (h+j)/2 – 1))$
$= floor\ (log\ (j–h–2))$
$\le floor\ (log\ (j–h))$
else $1 + floor\ (log\ (j – div\ (h+j)\ 2 – 1))$
$= floor\ (1 + log\ (j – (h+j–1)/2 – 1))$
$= floor\ (log\ (j–h–1))$
$\le floor\ (log\ (j–h))$ **fi**

The first refinement is

$(T \Longleftarrow h:= 0.\ j:= \#L.\ U)$ replace T and U

$= (t' \le t + floor\ (log\ (\#L)) \Longleftarrow h:= 0.\ j:= \#L.\ h<j \Rightarrow t' \le t + floor\ (log\ (j–h)))$

Substitution Law twice

$= (t' \le t + floor\ (log\ (\#L)) \Longleftarrow 0<\#L \Rightarrow t' \le t + floor\ (log\ (\#L–0)))$

given the list is nonempty, identity, and reflexive laws

$=\quad \top$

The middle refinement, starting with the right side, is

$$
\begin{array}{ll}
& i{:=}\ div\ (h{+}j)\ 2.\\
& \textbf{if}\ L\ i = x\ \textbf{then}\ p{:=}\ \top\\
& \textbf{else if}\ L\ i < x\ \textbf{then}\ h{:=}\ i{+}1\ \textbf{else}\ j{:=}\ i\ \textbf{fi}.\\
& \quad V\ \textbf{fi} \hfill \text{distribute last line back}\\[1ex]
= & i{:=}\ div\ (h{+}j)\ 2.\\
& \textbf{if}\ L\ i = x\ \textbf{then}\ p{:=}\ \top\\
& \textbf{else if}\ L\ i < x\ \textbf{then}\ h{:=}\ i{+}1.\ \ V\ \textbf{else}\ j{:=}\ i.\ \ V\ \textbf{fi fi} \quad \text{replace two } V\text{s and assignment to } p\\[1ex]
= & i{:=}\ div\ (h{+}j)\ 2.\\
& \textbf{if}\ L\ i = x\ \textbf{then}\ p' \wedge h'{=}h \wedge i'{=}i \wedge j'{=}j \wedge t'{=}t\\
& \textbf{else if}\ L\ i < x\\
& \quad \textbf{then}\ h{:=}\ i{+}1.\ \textbf{if}\ h{=}j\ \textbf{then}\ t'{=}t\ \textbf{else}\ h{<}j \Rightarrow t' \le t + 1 + floor(log(j{-}h))\ \textbf{fi}\\
& \quad \textbf{else}\ j{:=}\ i.\ \textbf{if}\ h{=}j\ \textbf{then}\ t'{=}t\ \textbf{else}\ h{<}j \Rightarrow t' \le t + 1 + floor(log(j{-}h))\ \textbf{fi fi fi}\\
& \hfill \text{substitution law for } h \text{ and } j\\
= & i{:=}\ div\ (h{+}j)\ 2.\\
& \textbf{if}\ L\ i = x\ \textbf{then}\ p' \wedge h'{=}h \wedge i'{=}i \wedge j'{=}j \wedge t'{=}t\\
& \textbf{else if}\ L\ i < x\\
& \quad \textbf{then if}\ i{+}1{=}j\ \textbf{then}\ t'{=}t\ \textbf{else}\ i{+}1{<}j \Rightarrow t' \le t + 1 + floor(log(j{-}i{-}1))\ \textbf{fi}\\
& \quad \textbf{else if}\ h{=}i\ \textbf{then}\ t'{=}t\ \textbf{else}\ h{<}i \Rightarrow t' \le t + 1 + floor(log(i{-}h))\ \textbf{fi fi fi}\\
& \hfill \text{substitution law for } i\\
= & \textbf{if}\ L(div\ (h{+}j)\ 2) = x\ \textbf{then}\ p' \wedge h'{=}h \wedge i'{=}div\ (h{+}j)\ 2 \wedge j'{=}j \wedge t'{=}t\\
& \textbf{else if}\ L(div\ (h{+}j)\ 2) < x\\
& \quad \textbf{then if}\ div\ (h{+}j)\ 2 + 1 = j\ \textbf{then}\ t'{=}t\\
& \qquad \textbf{else}\ div\ (h{+}j)\ 2 + 1 < j \Rightarrow t' \le t + 1 + floor(log(j - div\ (h{+}j)\ 2 - 1))\ \textbf{fi}\\
& \quad \textbf{else}\ \ \textbf{if}\ h = div\ (h{+}j)\ 2\ \textbf{then}\ t'{=}t\\
& \qquad \textbf{else}\ h < div\ (h{+}j)\ 2 \Rightarrow t' \le t + 1 + floor(log(div\ (h{+}j)\ 2 - h))\ \textbf{fi fi fi}\\
& \hfill \text{case analysis law}\\
= & \quad L(div(h{+}j)2) = x \wedge p' \wedge h'{=}h \wedge i'{=}div\ (h{+}j)\ 2 \wedge j'{=}j \wedge t'{=}t\\
& \vee\ \ L(div(h{+}j)2) < x \wedge div(h{+}j)2 + 1 = j \wedge t'{=}t\\
& \vee\ \ \quad L(div(h{+}j)2){<}x \wedge div(h{+}j)2 + 1 \ne j\\
& \quad \wedge\ \ (div(h{+}j)2 + 1 < j \Rightarrow t' \le t{+}1{+}floor(log(j - div(h{+}j)2 - 1)))\\
& \vee\ \ L(div(h{+}j)2){>}x \wedge h{=}div(h{+}j)2 \wedge t'{=}t\\
& \vee\ \ \quad L(div(h{+}j)2){>}x \wedge h{\ne}div(h{+}j)2\\
& \quad \wedge\ \ (h < div\ (h{+}j)\ 2 \Rightarrow t' \le t + 1 + floor(log(div\ (h{+}j)\ 2 - h)))
\end{array}
$$

We must prove that this disjunction implies the left side of the refinement U. By an antidistributive law, that is equivalent to proving that each of the three disjuncts implies the left side of the refinement. Since the left side of the refinement

$$h{<}j \Rightarrow t' \le t + floor\ (log\ (j{-}h))$$

is an implication, by portation, we prove that its antecedent $h{<}j$, and a disjunct, implies its consequent $t' \le t + floor\ (log\ (j{-}h))$. So, $h{<}j$ and first disjunct:

$$
\begin{array}{ll}
& h{<}j \wedge L(div(h{+}j)2) = x \wedge p' \wedge h'{=}h \wedge i'{=}div\ (h{+}j)\ 2 \wedge j'{=}j \wedge t'{=}t \hfill \text{specialize}\\
\Longrightarrow & h{<}j \wedge t'{=}t \hfill \text{arithmetic}\\
\Longrightarrow & t' \le t + floor\ (log\ (j{-}h))
\end{array}
$$

Second disjunct:

$$\begin{array}{lll} & h<j \wedge L(div(h+j)2) < x \wedge div(h+j)2 + 1 = j \wedge t'=t & \text{specialize} \\ \Longrightarrow & h<j \wedge t'=t & \text{arithmetic} \\ \Longrightarrow & t' \le t + floor\,(log\,(j-h)) & \end{array}$$

Next disjunct:

$$\begin{array}{lll} & \quad h<j \wedge L(div(h+j)2)<x \wedge div(h+j)2 + 1 \ne j & \\ & \wedge \quad (div(h+j)2 + 1 < j \Rightarrow t' \le t+1+floor(log(j - div(h+j)2 - 1))) & \text{specialize} \\ \Longrightarrow & \quad h<j \wedge div(h+j)2 + 1 \ne j & \\ & \wedge \quad (div(h+j)2 + 1 < j \Rightarrow t' \le t+1+floor(log(j - div(h+j)2 - 1))) & \text{Lemma A} \\ = & \quad h<j \wedge div\,(h+j)\,2 + 1 \le j \wedge div(h+j)2 + 1 \ne j & \\ & \wedge \quad (div(h+j)2 + 1 < j \Rightarrow t' \le t+1+floor(log(j - div(h+j)2 - 1))) & \text{combine } \le \text{ and } \ne \\ = & \quad h<j \wedge div\,(h+j)\,2 + 1 < j & \\ & \wedge \quad (div(h+j)2 + 1 < j \Rightarrow t' \le t+1+floor(log(j - div(h+j)2 - 1))) & \text{discharge} \\ = & h<j \wedge div\,(h+j)\,2 + 1 < j \quad \wedge \quad t' \le t+1+floor(log(j - div(h+j)2 - 1)) & \text{Lemma C} \\ \Longrightarrow & t' \le t + floor\,(log\,(j-h)) & \end{array}$$

Next disjunct:

$$\begin{array}{lll} & h<j \wedge L(div(h+j)2)>x \wedge h=div(h+j)2 \wedge t'=t & \text{specialize} \\ \Longrightarrow & h<j \wedge t'=t & \text{arithmetic} \\ \Longrightarrow & t' \le t + floor\,(log\,(j-h)) & \end{array}$$

Last disjunct:

$$\begin{array}{lll} & \quad h<j \wedge L(div(h+j)2)>x \wedge h\ne div(h+j)2 & \\ & \wedge \ (h < div\,(h+j)\,2 \Rightarrow t' \le t + 1 + floor(log(div\,(h+j)\,2 - h))) & \text{specialize} \\ \Longrightarrow & \quad h<j \wedge h\ne div(h+j)2 & \\ & \wedge \ (h < div\,(h+j)\,2 \Rightarrow t' \le t + 1 + floor(log(div\,(h+j)\,2 - h))) & \text{Lemma B} \\ \Longrightarrow & \quad h<j \wedge h \le div\,(h+j)\,2 \wedge h\ne div(h+j)2 & \\ & \wedge \ (h < div\,(h+j)\,2 \Rightarrow t' \le t + 1 + floor(log(div\,(h+j)\,2 - h))) & \text{combine } \le \text{ and } \ne \\ = & \quad h<j \wedge h < div\,(h+j)\,2 & \\ & \wedge \ (h < div\,(h+j)\,2 \Rightarrow t' \le t + 1 + floor(log(div\,(h+j)\,2 - h))) & \text{discharge} \\ = & h<j \wedge h < div\,(h+j)\,2 \ \wedge \ t' \le t + 1 + floor(log(div\,(h+j)\,2 - h)) & \text{Lemma C} \\ \Longrightarrow & t' \le t + floor\,(log\,(j-h)) & \end{array}$$

That completes the proof of the middle refinement.

Finally, we have to prove the last refinement.

$$\begin{array}{lll} & (V \ \Longleftarrow\ \textbf{if } h=j \textbf{ then } p:= \bot \textbf{ else } t:= t+1.\ U \textbf{ fi}) & \text{replace } V \\ = & (\textbf{if } h=j \textbf{ then } t'=t \textbf{ else } t:= t+1.\ U \textbf{ fi} \ \Longleftarrow\ \textbf{if } h=j \textbf{ then } p:= \bot \textbf{ else } t:= t+1.\ U \textbf{ fi}) & \\ & & \textbf{if-then-else-fi} \text{ is monotonic in its } \textbf{then}\text{-part.} \\ = & \top & \end{array}$$

(c) Find the execution time according to a measure that charges time 1 for each test.

§ $A \ \Longleftarrow\ h:= 0.\ j:= \#L.\ B$

$B \Leftarrow$ $i:= div\ (h+j)\ 2.\ \ t:= t+1.$ this time increment is for the test **if** $Li=x$
if $L\ i = x$ **then** $p:= \top$
else $t:= t+1.$ this time increment is for the test **if** $Li<x$
if $L\ i < x$ **then** $h:= i+1$ **else** $j:= i$ **fi**.
C **fi**

$C \Leftarrow$ $t:= t+1.$ this time increment is for the test **if** $h<j$
if $h=j$ **then** $p:= \bot$ **else** B **fi**

I know it's *log* time, but to get a good upper bound I tried a few list sizes by hand, with the value sought x less than all list values. So A , B , and C are defined as

$A \ = \ t' \le t + 3 + 3\times floor\ (log\ (\#L))$

$B \ = \ h<j \Rightarrow t' \le t + 3 + 3\times floor\ (log\ (j-h))$

$C \ = \ t:= t+1.\ \ \textbf{if}\ h=j\ \textbf{then}\ t'=t\ \textbf{else}\ B\ \textbf{fi}$

$\ \ = \ \textbf{if}\ t:= t+1.\ \ h=j\ \textbf{then}\ t:= t+1.\ \ t'=t\ \textbf{else}\ t:= t+1.\ \ B\ \textbf{fi}$

$\ \ = \ \textbf{if}\ h=j\ \textbf{then}\ t'\le t+1\ \textbf{else}\ h<j \Rightarrow t' \le t + 4 + 3\times floor\ (log\ (j-h))\ \textbf{fi}$

where *floor* is the function that rounds down.

These three refinements should now be proven, but I haven't got the energy.

(d) Compare the execution time to binary search without the test for equality each iteration.

§ The execution time bounds are:

Ex.187	recursive time, without equality test	$ceil\ (log\ (\#L))$
Ex.188(b)	recursive time, with equality test	$floor\ (log\ (\#L))$
Ex.188(c)	counting tests, with equality test	$3 + 3\times floor\ (log\ (\#L))$

Ex.187 and Ex.188(b) are almost the same; no significant difference.
Ex.188(c) takes about 3 times as long as Ex.187 and Ex.188(b).