200 (item count) Write a program to find the number of occurrences of a given item in a given list.

After trying the question, scroll down to the solution.

§ Let L be the list and a be the item. Let n and j be nat variables.

$$n' = \text{¢}(\S i\colon 0,..\#L\cdot L\,i = a) \;\Longleftarrow\; n:= 0.\; j:= 0.\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a)$$

$$n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) \;\Longleftarrow$$

$$\textbf{if } j=\#L \textbf{ then } ok \textbf{ else } \;\; \textbf{if } L\,j = a \textbf{ then } n:= n+1 \textbf{ else } ok \textbf{ fi}.\; j:= j+1.\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) \textbf{ fi}$$

The first refinement is proved by two uses of the Substitution Law. The last refinement can be broken into three cases. The first case:

$$\begin{array}{lll} & j=\#L \wedge ok & \text{expand } ok \\ = & j=\#L \wedge n'=n \wedge j'=j & \text{bunch theory and arithmetic} \\ \Longrightarrow & n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) & \end{array}$$

Middle case:

$$\begin{array}{lll} & j\ne\#L \wedge L\,j = a \wedge (n:= n+1.\; j:= j+1.\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a)) & \text{substitution twice} \\ = & j\ne A\#L \;\wedge\; Lj=a \;\wedge\; n' = n+1 + \text{¢}(\S i\colon j+1,..\#L\cdot L\,i = a) & \\ & & \text{if } L\,j = a \text{, then } \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) \;=\; 1 + \text{¢}(\S i\colon j+1,..\#L\cdot L\,i = a) \\ = & j\ne\#L \;\wedge\; L\,j = a \;\wedge\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) & \text{specialize} \\ \Longrightarrow & n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) & \end{array}$$

Last case:

$$\begin{array}{lll} & j\ne\#L \wedge L\,j \ne a \wedge (ok.\; j:= j+1.\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a)) & \\ & & \text{substitution and identity} \\ = & j\ne\#L \;\wedge\; Lj\ne a \;\wedge\; n' = n + \text{¢}(\S i\colon j+1,..\#L\cdot L\,i = a) & \\ & & \text{if } L\,j \ne a \text{, then } \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) \;=\; \text{¢}(\S i\colon j+1,..\#L\cdot L\,i = a) \\ = & j\ne\#L \;\wedge\; L\,j = a \;\wedge\; n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) & \text{specialize} \\ \Longrightarrow & n' = n + \text{¢}(\S i\colon j,..\#L\cdot L\,i = a) & \end{array}$$

Here are the timing refinements.

$$t' = t + \#L \;\Longleftarrow\; n:= 0.\; j:= 0.\; t' = t + \#L - j$$

$$t' = t + \#L - j \;\Longleftarrow$$

$$\textbf{if } j=\#L \textbf{ then } ok \textbf{ else } \;\; \textbf{if } L\,j = a \textbf{ then } n:= n+1 \textbf{ else } ok \textbf{ fi}.\; j:= j+1.\; t:= t+1.\; t' = t + \#L - j \textbf{ fi}$$

The first refinement is proven by the Substitution Law. The last breaks into three cases. The first case is:

$$\begin{array}{lll} & j=\#L \wedge ok & \text{expand } ok \\ = & j=\#L \wedge n'=n \wedge j'=j \wedge t'=t & \text{arithmetic} \\ \Longrightarrow & t' = t + \#L - j & \end{array}$$

Middle case:

$$\begin{array}{lll} & j\ne\#L \wedge Lj=a \wedge (n:= n+1.\; j:= j+1.\; t:= t+1.\; t' = t + \#L - j) & \text{substitution three times} \\ = & j\ne\#L \;\wedge\; L\,j = a \;\wedge\; t' = t + 1 + \#L - (j + 1) & \text{simplify and specialize} \\ \Longrightarrow & t' = t + \#L - j & \end{array}$$

Last case:

$$\begin{array}{lll} & j\ne\#L \wedge L\,j \ne a \wedge (ok.\; j:= j+1.\; t:= t+1.\; t' = t + \#L - j) & \text{substitution and identity} \\ = & j\ne\#L \;\wedge\; L\,j \ne a \;\wedge\; t' = t + 1 + \#L - (j + 1) & \text{simplify and specialize} \\ \Longrightarrow & t' = t + \#L - j & \end{array}$$

Here is a **for**-loop solution with invariant $A\,j \;=\; n = \text{¢}(\S i\colon 0,..j\cdot L\,i = a)$.

$$n' = \text{¢}(\S i\colon 0,..\#L\cdot L\,i = a) \;\Longleftarrow\; n:= 0.\; A\,0 \Rightarrow A'(\#L)$$

$$A\,0 \Rightarrow A'(\#L) \;\Longleftarrow\; \textbf{for } j:= 0;..\#L \textbf{ do } j\colon 0,..\#L \wedge A\,j \;\Rightarrow\; A'(j+1) \textbf{ od}$$

$$j\colon 0,..\#L \wedge A\,j \;\Rightarrow\; A'(j+1) \;\Longleftarrow\; \textbf{if } L\,j = a \textbf{ then } n:= n+1 \textbf{ else } ok \textbf{ fi}$$

To prove the first refinement, start with the right side.

$$\begin{array}{lll} & n:= 0.\; A\,0 \Rightarrow A'(\#L) & \text{expand } A \text{ and } A' \\ = & n:= 0.\; n = \text{¢}(\S i\colon 0,..0\cdot L\,i = a) \Rightarrow n' = \text{¢}(\S i\colon 0,..\#L\cdot L\,i = a) & \text{domain } 0,..0 \\ = & n:= 0.\; n = \text{¢}null \Rightarrow n' = \text{¢}(\S i\colon 0,..\#L\cdot L\,i = a) & \text{¢}null \\ = & n:= 0.\; n=0 \Rightarrow n' = \text{¢}(\S i\colon 0,..\#L\cdot L\,i = a) & \text{substitution law} \end{array}$$

$=\quad 0{=}0 \Rightarrow n' = ¢(§i: 0,..\#L \cdot L\,i = a)$ reflexive and identity

$=\quad n' = ¢(§i: 0,..\#L \cdot L\,i = a)$

The middle refinement is the invariant **for**-loop law. The last refinement is proven by cases. First:

$\quad L\,j = a \wedge (n{:=}\ n{+}1) \Rightarrow (j: 0,..\#L \wedge A\,j \Rightarrow A'(j{+}1))$ portation, expand assignment

$=\quad L\,j = a \wedge n'{=}n{+}1 \wedge j: 0,..\#L \wedge A\,j \Rightarrow A'(j{+}1)$ expand A and A'

$=\quad L\,j = a \wedge n'{=}n{+}1 \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢(§i: 0,..j{+}1 \cdot L\,i = a)$ split range $0,..j{+}1$

$=\quad L\,j = a \wedge n'{=}n{+}1 \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢((§i: 0,..j \cdot L\,i = a), (§i: j \cdot L\,i = a))$ the domains are disjoint

$=\quad L\,j = a \wedge n'{=}n{+}1 \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢(§i: 0,..j \cdot L\,i = a) + ¢(§i: j \cdot L\,i = a)$ context

$=\quad L\,j = a \wedge n'{=}n{+}1 \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a) \Rightarrow n'{=}n{+}1$ specialization

$=\quad \top$

Last case:

$\quad L\,j \ne a \wedge ok \Rightarrow (j: 0,..\#L \wedge A\,j \Rightarrow A'(j{+}1))$ portation, expand ok

$=\quad L\,j \ne a \wedge n'{=}n \wedge j: 0,..\#L \wedge A\,j \Rightarrow A'(j{+}1)$ expand A and A'

$=\quad L\,j \ne a \wedge n'{=}n \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢(§i: 0,..j{+}1 \cdot L\,i = a)$ split range $0,..j{+}1$

$=\quad L\,j \ne a \wedge n'{=}n \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢((§i: 0,..j \cdot L\,i = a), (§i: j \cdot L\,i = a))$ the domains are disjoint

$=\quad L\,j \ne a \wedge n'{=}n \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a)$
$\quad \Rightarrow n' = ¢(§i: 0,..j \cdot L\,i = a) + ¢(§i: j \cdot L\,i = a)$ context

$=\quad L\,j \ne a \wedge n'{=}n \wedge j: 0,..\#L \wedge n = ¢(§i: 0,..j \cdot L\,i = a) \Rightarrow n'{=}n{+}0$ specialization

$=\quad \top$

Time $\#L$.