204 (text length) You are given a text (string of characters) that begins with zero or more "ordinary" characters, and then ends with zero or more "padding" characters. A padding character is not an ordinary character. Write a program to find the number of ordinary characters in the text. Execution time should be logarithmic in the text length.

After trying the question, scroll down to the solution.

§ Let S be the text, let n be a natural variable to record the result. The problem is P where

$$P \;=\; (\forall i\colon 0,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..{\leftrightarrow}S\cdot\ \neg\ ord\ S_i)$$

We are given that such an n' exists and is unique. Let l be a natural variable. Define

$$Q \;=\; n{\le}l\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$$

Then the program is

$P \;\Leftarrow\; n{:=}\ 0.\ \ l{:=}\ {\leftrightarrow}S.\ \ Q$

$Q \;\Leftarrow\;$ **if** $n{=}l$ **then** ok
else $m{:=}\ div\ (n{+}l)\ 2.$
if $ord\ S_m$ **then** $n{:=}\ m{+}1$ **else** $l{:=}\ m$ **fi**.
Q **fi**

Proof of first refinement, starting with the right side:

$n{:=}\ 0.\ \ l{:=}\ {\leftrightarrow}S.\ \ Q$ — expand Q, then substitution twice

$=\ \ 0{\le}{\leftrightarrow}S\ \Rightarrow\ (\forall i\colon 0,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..{\leftrightarrow}S\cdot\ \neg\ ord\ S_i)$ — a length is nonnegative, and identity

$=\ \ P$

Before proving the last refinement, let me work on its **else**-part a little.

$m{:=}\ div\ (n{+}l)\ 2.$ **if** $ord\ S_m$ **then** $n{:=}\ m{+}1$ **else** $l{:=}\ m$ **fi**. Q — distribution

$=\ \ m{:=}\ div\ (n{+}l)\ 2.$ **if** $ord\ S_m$ **then** $n{:=}\ m{+}1.\ Q$ **else** $l{:=}\ m.\ Q$ **fi** — expand Q and use substitution law in both places

$=\ \ m{:=}\ div\ (n{+}l)\ 2.$
if $ord\ S_m$ **then** $m{+}1{\le}l\ \Rightarrow\ (\forall i\colon m{+}1,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$
else $n{\le}m\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..m\cdot\ \neg\ ord\ S_i)$ **fi** — substitution

$=\ $ **if** $ord\ S_{div\ (n+l)\ 2}$
then $div\ (n{+}l)\ 2\ {+}1 \le l\ \Rightarrow\ (\forall i\colon div\ (n{+}l)\ 2\ {+}1,..n'\cdot\ ord\ S_i) \land (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$
else $n \le div\ (n{+}l)\ 2\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..div\ (n{+}l)\ 2\cdot\ \neg\ ord\ S_i)$ **fi**

Now the last refinement looks like this:

Q
$\Leftarrow\ $ **if** $n{=}l$ **then** ok
else if $ord\ S_{div\ (n+l)\ 2}$
then $div\ (n{+}l)\ 2\ {+}1 \le l\ \Rightarrow\ (\forall i\colon div\ (n{+}l)\ 2\ {+}1,..n'\cdot\ ord\ S_i) \land (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$
else $n \le div\ (n{+}l)\ 2\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..div\ (n{+}l)\ 2\cdot\ \neg\ ord\ S_i)$ **fi fi**

It can now be proved in three cases. First case:

$Q\ \Leftarrow\ n{=}l \land ok$ — expand Q and ok

$=\ \ (n{\le}l\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i))$
$\Leftarrow\ n{=}l \land n'{=}n \land l'{=}l \land m'{=}m$ — portation

$=\ \ n{\le}l \land n{=}l \land n'{=}n \land l'{=}l \land m'{=}m\ \Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$ — context

$=\ \ n{\le}l \land n{=}l \land n'{=}n \land l'{=}l \land m'{=}m\ \Rightarrow\ (\forall i\colon n,..n\cdot\ ord\ S_i)\ \land\ (\forall i\colon n,..n\cdot\ \neg\ ord\ S_i)$ — quantifier law about empty domain

$=\ \ n{\le}l \land n{=}l \land n'{=}n \land l'{=}l \land m'{=}m\ \Rightarrow\ \top \land \top$ — idempotent and base

$=\ \ \top$

Last refinement, middle case:

$Q\ \Leftarrow\ \ n{\ne}l \land ord\ S_{div\ (n+l)\ 2}$
$\land\ (\ \ div\ (n{+}l)\ 2\ {+}1 \le l$
$\Rightarrow\ (\forall i\colon div\ (n{+}l)\ 2\ {+}1,..n'\cdot\ ord\ S_i) \land (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i))$ — expand Q and portation

$=\ \ \ n{\le}l \land n{\ne}l \land ord\ S_{div\ (n+l)\ 2}$
$\land\ (div\ (n{+}l)\ 2\ {+}1 \le l \Rightarrow (\forall i\colon div\ (n{+}l)\ 2\ {+}1,..n'\cdot\ ord\ S_i) \land (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i))$
$\Rightarrow\ (\forall i\colon n,..n'\cdot\ ord\ S_i)\ \land\ (\forall i\colon n',..l\cdot\ \neg\ ord\ S_i)$

We have $n{\le}l \land n{\ne}l$, which equals $n{<}l$, which implies $div\ (n{+}l)\ 2\ {+}1 \le l$, and that discharges the antecedent of the first implication.

$=$ $\quad n<l \land ord\ S_{div\ (n+l)\ 2} \land (\forall i{:}\ div\ (n+l)\ 2+1,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..l\cdot\ \neg\ ord\ S_i)$
$\Rightarrow (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..l\cdot\ \neg\ ord\ S_i)$ From the given information and from $ord\ S_{div\ (n+l)\ 2}$ we get $(\forall i{:}\ n,..div\ (n+l)\ 2+1\cdot\ ord\ S_i)$. From that and $(\forall i{:}\ div\ (n+l)\ 2+1,..n'\cdot\ ord\ S_i)$ we get $(\forall i{:}\ n,..n'\cdot\ ord\ S_i)$.

$=$ $\top$

Last refinement, last case:

$Q \Leftarrow \quad n \neq l \land \neg\ ord\ S_{div\ (n+l)\ 2}$
$\land (n \le div\ (n+l)\ 2 \Rightarrow (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..div\ (n+l)\ 2\cdot\ \neg\ ord\ S_i))$ expand Q and portation

$=$ $\quad n \le l \land n \neq l \land \neg\ ord\ S_{div\ (n+l)\ 2}$
$\land (n \le div\ (n+l)\ 2 \Rightarrow (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..div\ (n+l)\ 2\cdot\ \neg\ ord\ S_i))$
$\Rightarrow (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..l\cdot\ \neg\ ord\ S_i)$
We have $n \le l$, which implies $n \le div\ (n+l)\ 2$, and that discharges the antecedent of the first implication.

$=$ $\quad n<l \land \neg\ ord\ S_{div\ (n+l)\ 2} \land (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..div\ (n+l)\ 2\cdot\ \neg\ ord\ S_i))$
$\Rightarrow (\forall i{:}\ n,..n'\cdot\ ord\ S_i) \land (\forall i{:}\ n',..l\cdot\ \neg\ ord\ S_i)$ From the given information and from $\neg\ ord\ S_{div\ (n+l)\ 2}$ we get $(\forall i{:}\ div\ (n+l)\ 2,..l\cdot\ \neg\ ord\ S_i)$. From that and $(\forall i{:}\ n',..div\ (n+l)\ 2\cdot\ \neg\ ord\ S_i))$ we get $(\forall i{:}\ n',..l\cdot\ \neg\ ord\ S_i)$.

$=$ $\top$

Timing: replace P with **if** $\leftrightarrow S=0$ **then** $t'=t$ **else** $t' \le t+1+log(\leftrightarrow S)$ **fi** and replace Q with $(n=l \Rightarrow t'=t) \land (n<l \Rightarrow t' \le t+1+log(l-n))$.