219 (natural division) The natural quotient of natural $n$ and positive integer $p$ is the natural number $q$ satisfying

$$q \le n/p < q+1$$

Write a program to find the natural quotient of $n$ and $p$ in $\log n$ time without using functions *div* , *mod* , *floor* , or *ceil* .

After trying the question, scroll down to the solution.

§ Natural $n$ and positive integer $p$ are given constants. Let $q$ be a natural variable. The specification is $S$, defined as

$$S \;=\; q' \le n/p < q'+1$$

Since $0 \le q' \le n$ we can do a binary search for $q'$ and thus achieve $log\ n$ time. Modeling the solution on Subsection 4.2.5, let $r$ and $s$ be natural variables. Define specification $R$ as

$$R \;=\; q \le n/p < s+1 \;\Rightarrow\; q \le q' \le n/p < q'+1 \le s+1$$

Now refine.

$$S \;\Leftarrow\; q{:=}\,0.\;\; s{:=}\,n.\;\; R$$

$$R \;\Leftarrow\; \textbf{if } q{=}s \textbf{ then } ok \textbf{ else } r{:=}\,div\ (q{+}s)\ 2.\;\; \textbf{if } r{\times}p \le n \textbf{ then } q{:=}\,r.\;\; R \textbf{ else } s{:=}\,r{-}1.\;\; R \textbf{ fi fi}$$

I have used $div$, contrary to instructions. But it's easy to get rid of it.

$$div\ (q{+}s)\ 2 \;=\; \textbf{if } even\ (q{+}s) \textbf{ then } (q{+}s)/2 \textbf{ else } (q{+}s{-}1)/2 \textbf{ fi}$$

We can get rid of $div\ (q{+}s)\ 2$ with $2$ cases. If we had used $div\ n\ p$, we wouldn't be able to get rid of it because we can't make $p$ cases for arbitrary $p$.

The solution is missing the proofs.