227 (maximum product segment) Given a list of integers, possibly including negatives, write a program to find

(a) the maximum product of any segment (sublist of consecutive items).

(b) the segment whose product is maximum.

After trying the question, scroll down to the solution.

(a) the maximum product of any segment (sublist of consecutive items).

§ The problem is P , defined as

$$P \;=\; p' = \Uparrow i\colon 0,..\#L+1\cdot\ \Uparrow j\colon i,..\#L+1\cdot\ \Pi\, L\,[i;..j]$$

using *int* variable p for the answer. We also use variable k: *nat* as a list index, and variables c, d, x: *int* . Define

$$\begin{aligned} J \;=\;\; & p = (\Uparrow i\colon 0,..k+1\cdot\ \Uparrow j\colon i,..k+1\cdot\ \Pi\, L\,[i;..j]) \\ \wedge\;\; & c = (\Uparrow i\colon 0,..k+1\cdot\ \Pi\, L\,[i;..k]) \\ \wedge\;\; & d = (\Downarrow i\colon 0,..k+1\cdot\ \Pi\, L\,[i;..k]) \end{aligned}$$

Here are the refinements.

$$P \;\Leftarrow\; p{:=}\,1.\ c{:=}\,1.\ d{:=}\,1.\ k{:=}\,0.\ J{\Rightarrow}P$$

$$\begin{aligned} J{\Rightarrow}P \;\Leftarrow\;\; & \textbf{if } k{=}\#L \textbf{ then } ok \\ & \textbf{else}\ \ \textbf{if } L\,k \ge 0 \textbf{ then } c{:=}\,(c \times L\,k){\uparrow}1.\ d{:=}\,(d \times L\,k){\downarrow}1 \\ & \qquad\ \textbf{else } x{:=}\,c.\ c{:=}\,(d \times L\,k){\uparrow}1.\ d{:=}\,(x \times L\,k){\downarrow}1 \textbf{ fi}. \\ & \qquad\ p{:=}\,p{\uparrow}c.\ k{:=}\,k{+}1.\ J{\Rightarrow}P \textbf{ fi} \end{aligned}$$

and the timing is

$$t' = t{+}\#L \;\Leftarrow\; p{:=}\,1.\ c{:=}\,1.\ d{:=}\,1.\ k{:=}\,0.\ t' = t{+}\#L{-}k$$

$$\begin{aligned} t' = t{+}\#L{-}k \;\Leftarrow\;\; & \textbf{if } k{=}\#L \textbf{ then } ok \\ & \textbf{else}\ \ \textbf{if } L\,k \ge 0 \textbf{ then } c{:=}\,(c \times L\,k){\uparrow}1.\ d{:=}\,(d \times L\,k){\downarrow}1 \\ & \qquad\ \textbf{else } x{:=}\,c.\ c{:=}\,(d \times L\,k){\uparrow}1.\ d{:=}\,(x \times L\,k){\downarrow}1 \textbf{ fi}. \\ & \qquad\ p{:=}\,p{\uparrow}c.\ k{:=}\,k{+}1.\ t{:=}\,t{+}1.\ t' = t{+}\#L{-}k \textbf{ fi} \end{aligned}$$

Proof of the first refinement: after 4 substitutions, J simplifies to $\top$.

The second refinement breaks into 3 cases. Each case begins with portation, so we are proving

$$J \wedge \text{something} \;\Rightarrow\; P$$

by starting with the antecedent. First case:

$$\begin{aligned} & J \wedge k{=}\#L \wedge ok && \text{in context } k{=}\#L \wedge p'{=}p \text{ the first conjunct of } J \text{ is } P \\ \Rightarrow\;\; & P \end{aligned}$$

Second case:

$$\begin{aligned} & J \wedge k{\ne}\#L \wedge L\,k \ge 0 \\ & \wedge\ (c{:=}\,(c \times L\,k){\uparrow}1.\ d{:=}\,(d \times L\,k){\downarrow}1).\ p{:=}\,p{\uparrow}c.\ k{:=}\,k{+}1.\ J{\Rightarrow}P) \end{aligned}$$

Make 4 substitutions. Note that P does not mention any of the 4 variables (it mentions p' but not p).

I'm reversing $J{\Rightarrow}P$ to $P{\Leftarrow}J$ for typesetting reasons.

$$\begin{aligned} =\;\; & J \wedge k{\ne}\#L \wedge L\,k \ge 0 \\ & \wedge\ (P \Leftarrow (\quad p{\uparrow}(c \times Lk){\uparrow}1 = (\Uparrow i\colon 0,..k+2\cdot\ \Uparrow j\colon i,..k+2\cdot\ \Pi\, L\,[i;..j]) \\ & \qquad\qquad \wedge\ (c \times L\,k){\uparrow}1 = (\Uparrow i\colon 0,..k+2\cdot\ \Pi\, L\,[i;..k+1]) \\ & \qquad\qquad \wedge\ (d \times L\,k){\downarrow}1 \;=\; (\Downarrow i\colon 0,..k+2\cdot\ \Pi\, L\,[i;..k+1]))) \end{aligned}$$

We need $J \wedge k{\ne}\#L \wedge L\,k \ge 0$ to discharge the implication, so we need to show that it implies the antecedent of the implication. J says that p is the maximum product of all segments ending at or before k , and that c is the maximum product of all segments ending at k , and that d is the minimum product of all segments ending at k (remember that we write indexes between items). To find the maximum product of all segments ending at or before $k+1$, we need only consider the new sequences, which are those ending at $k+1$. One of them is the empty sequence whose product is 1 . The others are all one-item extensions of sequences ending at k . Since the new item $L\,k$ is nonnegative, the maximum product of these extensions is the maximum product c of those sequences ending at k times the new item $L\,k$. So the maximum product of all segments ending at or before k is the maximum of p , $c \times L\,k$, and 1 . That's what the first two conjuncts of the antecedent say, so they are discharged. The last conjunct of the antecedent is also discharged by a similar argument. This hint is ridiculously long, informal, and inadequate.

$\Longrightarrow \quad P$

The last case is much like the previous case, but slightly more complicated because $L\,k$ is negative, and so multiplying by it switches maximums and minimums.

(b) the segment (sublist of consecutive items) whose product is maximum.

no solution given