248 (transitive closure) A relation $R: (0,..n)\to(0,..n)\to bin$ can be represented by a square binary array of size n. Given a relation in the form of a square binary array, write a program to find

(a) its transitive closure (the strongest transitive relation that is implied by the given relation).

§ Let $P\ i\ j\ k$ mean "there is a path in R from j to k via zero or more intermediate nodes all of which are less than i". Formally,

$$P\ 0 = R$$
$$\forall i, j, k\cdot P(i+1)j\ k \;=\; P\ i\ j\ k \lor P\ i\ j\ i \land P\ i\ i\ k$$

Then we can say that R' is the transitive closure of R as follows:

$$R' = P\ n$$

This is just right for a **for**-loop (Chapter 5) with invariant $R = P\ i$.

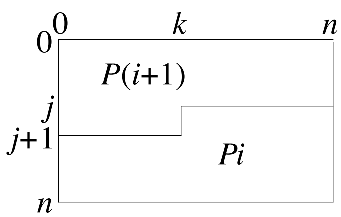
$R = P\ 0 \Rightarrow R' = P\ n \;\Longleftarrow\;$ **for** i:= 0;..n **do** $R = P\ i \Rightarrow R'=P(i+1)$ **od**

$R = P\ i \Rightarrow R'=P(i+1) \;\Longleftarrow$

  for j:= 0;..n **do for** k:= 0;..n **do** R:= $(j;k) \to R\ j\ k \lor R\ j\ i \land R\ i\ k \mid R$ **od od**

That's the whole thing. If you want more detail, define A as follows.

$$\begin{aligned} A\ i\ j\ k \;=\;& (\forall r: 0,..j\cdot \forall c: 0,..n\cdot R\ r\ c = P(i+1)r\ c)\\ \land\;& (\forall c: 0,..k\cdot R\ j\ c = P(i+1)j\ c)\\ \land\;& (\forall c: k,..n\cdot R\ j\ c = P\ i\ j\ c)\\ \land\;& (\forall r: j+1,..n\cdot \forall c: k,..n\cdot R\ r\ c = P\ i\ r\ c)\end{aligned}$$



Now $A\ i\ 0\ 0 \;=\; R = P\ i$ and $A\ i\ j\ n \;=\; A\ i(j+1)0$ and $A\ i\ n\ 0 \;=\; A(i+1)0\ 0$.

$A\ 0\ 0\ 0 \Rightarrow A'n\ 0\ 0 \;\Longleftarrow\;$ **for** i:= 0;..n **do** $A\ i\ 0\ 0 \Rightarrow A'(i+1)0\ 0$ **od**

$A\ i\ 0\ 0 \Rightarrow A'(i+1)0\ 0 \;\Longleftarrow\;$ **for** j:= 0;..n **do** $A\ i\ j\ 0 \Rightarrow A'i(j+1)0$ **od**

$A\ i\ j\ 0 \Rightarrow A'i(j+1)0 \;\Longleftarrow\;$ **for** k:= 0;..n **do** $A\ i\ j\ k \Rightarrow A'i\ j(k+1)$ **od**

$A\ i\ j\ k \Rightarrow A'i\ j(k+1) \;\Longleftarrow\; R$:= $(j;k) \to R\ j\ k \lor R\ j\ i \land R\ i\ k$

Of course, **for**-loops are not necessary.

(b) its reflexive transitive closure (the strongest reflexive and transitive relation that is implied by the given relation).

§ This is similar to part (a), but this time we define $P\ 0\ j\ k \;=\; j{=}k \lor R\ j\ k$. Since $P\ 0$ is not true initially, we need to start with

$R' = P\ n \;\Longleftarrow\;$ **for** j:= 0;..n **do** R:= $(j;j) \to \top \mid R$ **od**.

  $R = P\ 0 \Rightarrow R' = P\ n$

and then continue as before.