265 (ultimately periodic sequence) You are given function f: $int \rightarrow int$ such that the sequence

$$x_0 = 0$$
$$x_{n+1} = f(x_n)$$

generated by f starting at 0 is ultimately periodic:

$$\exists p: nat+1 \cdot \exists n: nat \cdot x_n = x_{n+p}$$

The smallest positive p such that $\exists n: nat \cdot x_n = x_{n+p}$ is called the period. Write a program to find the period. Your program should use an amount of storage that is bounded by a constant, and not dependent on f.

After trying the question, scroll down to the solution.

§ An ultimately periodic sequence is like a figure 6 . We will report the result in natural variable p . The specification is P , defined as

$$P \;=\; p' = \Downarrow p\text{: } (\S p\text{: } nat{+}1 \cdot \exists n\text{: } nat \cdot x_n = x_{n+p}) \cdot p$$

or

$$P \;=\; 1{\le}p' \wedge \exists n\text{: } nat \cdot x_n{=}x_{n+p'} \wedge \forall q\text{: } 1,..p' \cdot x_n{\ne}x_{n+q}$$

The given information, that the sequence is ultimately periodic, is necessary so that P is implementable. (If the sequence were not ultimately periodic, we would have to let p be an extended natural so that P is implementable.) Let a and b be integer variables used for sequence items, and let n be a natural variable used as a sequence index.

Let $Q \;=\; b{=}x_{2\times n+1} \Rightarrow P$

and $R \;=\; a{=}x_n \wedge b{=}x_{n+p} \wedge p{\le}n \wedge (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k}) \;\Rightarrow\; P$

The refinements are

$$P \Leftarrow b{:=}f\,0.\;\; n{:=}\,0.\;\; Q$$

$$Q \Leftarrow a{:=}\,b.\;\; n{:=}\,2{\times}n{+}1.\;\; p{:=}\,0.\;\; R$$

$R \Leftarrow b{:=}f\,b.\;\; p{:=}\,p{+}1.$ **if** $a{=}b$ **then** ok **else if** $p{\le}n$ **then** R **else** Q **fi fi**

Proof of first refinement:

$b{:=}f\,0.\;\; n{:=}\,0.\;\; Q$ — expand Q and 2 substitutions

$=\;\; f\,0 = x_1 \;\Rightarrow\; P$ — use given information

$=\;\; P$

Second refinement:

$a{:=}\,b.\;\; n{:=}\,2{\times}n{+}1.\;\; p{:=}\,0.\;\; R$ — expand R and 3 substitutions

$=\;\; b{=}x_{2\times n+1} \;\wedge\; b{=}x_{2\times n+1} \;\wedge\; 0{\le}2{\times}n{+}1 \;\wedge\; (\forall k\text{: } 0,..0 \cdot x_{2\times n+1}{\ne}x_{2\times n+2+k}) \;\Rightarrow\; P$

$=\;\; b{=}x_{2\times n+1} \;\Rightarrow\; P$

$=\;\; Q$

Last refinement, first case:

$(R \;\Leftarrow\; b{:=}f\,b.\;\; p{:=}\,p{+}1.\;\; a{=}b \wedge ok)$ — expand R

$=\;\; (\quad (a{=}x_n \;\wedge\; b{=}x_{n+p} \;\wedge\; p{\le}n \;\wedge\; (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k}) \;\Rightarrow\; P)$

$\Leftarrow\; b{:=}f\,b.\;\; p{:=}\,p{+}1.\;\; a{=}b \wedge ok)$ — portation

$=\;\; (b{:=}f\,b.\;\; p{:=}\,p{+}1.\;\; a{=}b \wedge ok) \wedge a{=}x_n \wedge b{=}x_{n+p} \wedge p{\le}n \wedge (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k})$

$\Rightarrow\; P$ — expand P ; also expand ok and make 2 substitutions

$=\;\; a = f\,b = a' = b' \wedge n'{=}n \wedge p'{=}p{+}1 \wedge a{=}x_n \wedge b{=}x_{n+p} \wedge p{\le}n \wedge (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k})$

$\Rightarrow\; 1{\le}p' \wedge (\exists n\text{: } nat \cdot x_n{=}x_{n+p'} \wedge \forall q\text{: } 1,..p' \cdot x_n{\ne}x_{n+q})$

From $p'{=}p{+}1$ we have $1{\le}p'$.

From $a = f\,b$ and $a{=}x_n$ and $b{=}x_{n+p}$ and $p'{=}p{+}1$ we have $x_n{=}x_{n+p'}$.

From $\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k}$ and $p'{=}p{+}1$ we have $\forall q\text{: } 1,..p' \cdot x_n{\ne}x_{n+q}$.

$=\;\; \top$

Last refinement, second case:

$(R \;\Leftarrow\; b{:=}f\,b.\;\; p{:=}\,p{+}1.\;\; a{\ne}b \wedge p{\le}n \wedge R)$ — expand last R and 2 substitutions

$=\;\; (\quad R$

$\Leftarrow\;\; a \ne f\,b \wedge p{+}1{\le}n$

$\wedge\; (a{=}x_n \wedge f\,b = x_{n+p+1} \wedge p{+}1{\le}n \wedge (\forall k\text{: } 0,..p{+}1 \cdot x_n{\ne}x_{n+1+k}) \;\Rightarrow\; P))$

expand R , and since it becomes an implication, use portation

$=\;\; a \ne f\,b \wedge p{+}1{\le}n \wedge (a{=}x_n \wedge f\,b = x_{n+p+1} \wedge p{+}1{\le}n \wedge (\forall k\text{: } 0,..p{+}1 \cdot x_n{\ne}x_{n+1+k}) \;\Rightarrow\; P)$

$\wedge\; a{=}x_n \wedge b{=}x_{n+p} \wedge p{\le}n \wedge (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k})$

$\Rightarrow\; P$ — the inner implication is discharged by the other conjuncts

$=\;\; a \ne f\,b \wedge p{+}1{\le}n \wedge P \wedge a{=}x_n \wedge b{=}x_{n+p} \wedge p{\le}n \wedge (\forall k\text{: } 0,..p \cdot x_n{\ne}x_{n+1+k})$

$\Rightarrow\; P$ — specialize

$=\;\; \top$

Last refinement, last case:

$(R \;\Leftarrow\; b{:=}f\,b.\;\; p{:=}\,p{+}1.\;\; a{\ne}b \wedge p{>}n \wedge Q)$ — expand Q and 2 substitutions

$=\;\; (R \;\Leftarrow\; a \ne f\,b \wedge p{+}1{>}n \wedge (f\,b = x_{2\times n+1} \;\Rightarrow\; P))$

expand R , and since it becomes an implication, use portation

$=\quad a \neq f\,b \wedge p+1>n \wedge (f\,b = x_{2\times n+1} \Rightarrow P) \wedge a=x_n \wedge b=x_{n+p} \wedge p\le n \wedge (\forall k\colon 0,..p\cdot\ x_n \neq x_{n+1+k})$
$\quad \Rightarrow\ P$

drop irrelevant conjuncts from antecedent, and put $p+1>n$ and $p\le n$ together

$\Leftarrow\quad p=n \wedge b=x_{n+p} \wedge (f\,b = x_{2\times n+1} \Rightarrow P)\ \Rightarrow\ P$ discharge

$=\quad p=n \wedge b=x_{n+p} \wedge P\ \Rightarrow\ P$ specialize

$=\quad \top$

The timing is:

$t' = t+n'+p'-1 \ \Leftarrow\ b:= f\,0.\ n:= 0.\ t' = t+n'+p'-2\times n-1$

$t' = t+n'+p'-2\times n-1 \ \Leftarrow\ a:= b.\ n:= 2\times n+1.\ p:= 0.\ p\le n \Rightarrow t' = t+n'+p'-n-p$

$p\le n \Rightarrow t' = t+n'+p'-n-p \ \Leftarrow\ b:= f\,b.\ p:= p+1.\ t:= t+1.$
if $a=b$ **then** ok
else if $p\le n$ **then** $p\le n \Rightarrow t' = t+n'+p'-n-p$
else $t' = t+n'+p'-2\times n-1$ **fi fi**

It is not very useful to give the time as $n'+p'-1$, but that's all that can be said. Here are the proofs. First refinement:

$b:= f\,0.\ n:= 0.\ t' = t+n'+p'-2\times n-1$ substitution

$=\quad t' = t+n'+p'-1$

Second refinement:

$a:= b.\ n:= 2\times n+1.\ p:= 0.\ p\le n \Rightarrow t' = t+n'+p'-n-p$ substitutions

$=\quad t' = t+n'+p'-2\times n-1$

Last refinement, first case:

$b:= f\,b.\ p:= p+1.\ t:= t+1.\ a=b \wedge ok$ expand ok and drop useless conjuncts

$\Longrightarrow\quad b:= f\,b.\ p:= p+1.\ t:= t+1.\ t'=t \wedge n'=n \wedge p'=p$ substitutions

$=\quad t'=t+1 \wedge n'=n \wedge p'=p+1$

$\Longrightarrow\quad p\le n \Rightarrow t' = t+n'+p'-n-p$

Last refinement, second case:

$b:= f\,b.\ p:= p+1.\ t:= t+1.\ a\neq b \wedge p\le n \wedge (p\le n \Rightarrow t' = t+n'+p'-n-p)$ discharge, then drop $a\neq b \wedge p\le n$

$\Longrightarrow\quad b:= f\,b.\ p:= p+1.\ t:= t+1.\ t' = t+n'+p'-n-p$ substitutions

$=\quad t' = t+n'+p'-n-p$

$\Longrightarrow\quad p\le n \Rightarrow t' = t+n'+p'-n-p$

Last refinement, last case:

$(\quad p\le n \Rightarrow t' = t+n'+p'-n-p$
$\Leftarrow\ b:= f\,b.\ p:= p+1.\ t:= t+1.\ a\neq b \wedge p>n \wedge t' = t+n'+p'-2\times n-1)$ substitutions

$=\quad (\quad p\le n \Rightarrow t' = t+n'+p'-n-p$
$\Leftarrow\ a\neq fb \wedge p+1>n \wedge t' = t+1+n'+p'-2\times n-1)$ the 1s cancel; portation

$=\quad a\neq fb \wedge p+1>n \wedge t' = t+n'+p'-2\times n \wedge p\le n \Rightarrow t' = t+n'+p'-n-p$

from $p+1>n$ and $p\le n$ we have $p=n$; then specialize

$=\quad \top$