274 (smallest common item) Given two sorted lists having at least one item in common, write a program to find the smallest item occurring in both lists.

After trying the question, scroll down to the solution.

§ Let the two lists be A and B, and let a and b be natural variables to act as indexes of the two lists. The specification is R, defined as

$$\begin{aligned} R \;=\;\; & A(0,..a') \text{ ‘ } B(0,..b') = null \\ & \land\; A\,a' = B\,b \\ & \land\; t' = t+a'+b' \end{aligned}$$

which says there isn't a common item before indexes a' and b', and there is one at those indexes, and the sum of those indexes is the execution time. We also need the specification Q, defined as

$$\begin{aligned} Q \;=\;\; & A(a,..a') \text{ ‘ } B(b,..b') = null \\ & \land\; A\,a' = B\,b' \\ & \land\; t' = t+a'-a+b'-b \end{aligned}$$

which says the same thing, but starting at indexes a and b. The refinements are

$R \;\Leftarrow\; a{:=}\,0.\; b{:=}\,0.\; Q$

$Q \;\Leftarrow\;$ **if** $A\,a < B\,b$ **then** $a{:=}\,a+1.\; t{:=}\,t+1.\; Q$
  else if $B\,b < C\,c$ **then** $b{:=}\,b+1.\; t{:=}\,t+1.\; Q$
  else ok **fi fi**

The proof of the first refinement is immediate after using the Substitution Law 2 times. The proof of the last refinement breaks into 9 pieces (3 conjuncts × 3 cases). Let's start with the first conjunct of Q and the first case of the **if**.

$A\,a < B\,b \land (a{:=}\,a+1.\; t{:=}\,t+1.\; A(a,..a') \text{ ‘ } B(b,..b') = null)$ substitution law twice

$=\;\; A\,a < B\,b \;\land\; A(a+1,..a') \text{ ‘ } B(b,..b') = null$

Because B is sorted, $B\,b$ is the minimum item of $B(b,..b')$.
And since $A\,a < B\,b$, therefore $A\,a$ is unequal to any item in $B(b,..b')$.

$\Rightarrow\;\; A(a,..a') \text{ ‘ } B(b,..b') = null$

Now we prove the middle conjunct of Q with the same first case of the **if**.

$A\,a < B\,b \land (a{:=}\,a+1.\; t{:=}\,t+1.\; A\,a' = B\,b')$ substitution law twice

$=\;\; A\,a < B\,b \;\land\; A\,a' = B\,b'$ specialize

$\Rightarrow\;\; A\,a' = B\,b'$

Now we prove the last conjunct of Q with the same first case of the **if**.

$A\,a < B\,b \land (a{:=}\,a+1.\; t{:=}\,t+1.\; t' = t+a'-a+b'-b)$ substitution law twice

$=\;\; A\,a < B\,b \;\land\; t' = t+1+a'-a-1+b'-b$ simplify and specialize

$\Rightarrow\;\; t' = t+a'-a+b'-b$

The proof of the middle case is exactly the same. That leaves the last case.

$A\,a \ge B\,b \land B\,b \ge A\,a \land ok$ antisymmetry, and expand ok

$A\,a = B\,b \;\land\; a'=a \;\land\; b'=b \;\land\; t'=t$

$\Rightarrow\;\; Q$