275 (longest common prefix) A natural number can be written as a sequence of decimal digits with a single leading zero. Given two natural numbers, write a program to find the number that is written as their longest common prefix of digits. For example, given 025621 and 02547 , the result is 025 . Hint: this question is about numbers, not about strings or lists.

After trying the question, scroll down to the solution.

§ This question resembles the greatest common divisor Exercise 270. Let $lcp\ a\ b$ be the number that is written as the longest common prefix of a and b. The only properties of lcp that we need are

(a) $lcp\ a\ a\ =\ a$

(b) $a{>}b\ \ \Rightarrow\ \ lcp\ a\ b = lcp\ (div\ a\ 10)\ b$

(c) $a{<}b\ \ \Rightarrow\ \ lcp\ a\ b = lcp\ a\ (div\ b\ 10)$

In fact, those three properties are sufficient to define lcp.

$$a' = b' = lcp\ a\ b\ \ \Leftarrow\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ ok$$
$$\mathbf{else\,if}\ a{>}b\ \mathbf{then}\ a{:=}\ div\ a\ 10.\ \ a' = b' = lcp\ a\ b$$
$$\mathbf{else}\ b{:=}\ div\ b\ 10.\ \ a' = b' = lcp\ a\ b\ \mathbf{fi\ fi}$$

Proof of first case:

$$a{=}b \land ok\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b \qquad \text{expand } ok$$
$$= \quad a{=}b \land a'{=}a \land b'{=}b\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b \qquad \text{context}$$
$$= \quad a{=}b \land a'{=}a \land b'{=}b\ \ \Rightarrow\ \ a = a = lcp\ a\ a \qquad \text{reflexive, property (a)}$$
$$= \quad a{=}b \land a'{=}a \land b'{=}b\ \ \Rightarrow\ \ \top \qquad \text{base}$$
$$= \quad \top$$

Proof of middle case:

$$a{\ne}b \land a{>}b \land (a{:=}\ div\ a\ 10.\ \ a' = b' = lcp\ a\ b)\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b$$
simplify and substitution law
$$= \quad a{>}b\ \land\ a' = b' = lcp\ (div\ a\ 10)\ b\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b \qquad \text{property (b)}$$
$$= \quad \top$$

Proof of last case:

$$a{\ne}b \land a{\le}b \land (b{:=}\ div\ b\ 10.\ \ a' = b' = lcp\ a\ b)\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b$$
simplify and substitution law
$$= \quad a{<}b\ \land\ a' = b' = lcp\ a\ (div\ b\ 10)\ \ \Rightarrow\ \ a' = b' = lcp\ a\ b \qquad \text{property (c)}$$
$$= \quad \top$$

Now for the time.

$$t' \le t{+}a{+}b\ \ \Leftarrow\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ ok$$
$$\mathbf{else\ if}\ a{>}b\ \mathbf{then}\ a{:=}\ div\ a\ 10.\ \ t{:=}\ t{+}1.\ \ t' \le t{+}a{+}b$$
$$\mathbf{else}\ b{:=}\ div\ b\ 10.\ \ t{:=}\ t{+}1.\ \ t' \le t{+}a{+}b\ \mathbf{fi\ fi}$$

Proof of first case:

$$a{=}b \land ok\ \ \Rightarrow\ \ t' \le t{+}a{+}b \qquad \text{expand } ok$$
$$= \quad a{=}b \land a'{=}a \land b'{=}b \land t'{=}t\ \ \Rightarrow\ \ t' \le t{+}a{+}b \qquad a \text{ and } b \text{ are natural}$$
$$= \quad \top$$

Proof of middle case:

$$a{\ne}b \land a{>}b \land (a{:=}\ div\ a\ 10.\ \ t{:=}\ t{+}1.\ \ t' \le t{+}a{+}b)\ \ \Rightarrow\ \ t' \le t{+}a{+}b$$
simplify and substitution law twice
$$= \quad a{>}b\ \land\ t' \le t{+}1{+}(div\ a\ 10){+}b\ \ \Rightarrow\ \ t' \le t{+}a{+}b \qquad \text{connection}$$
$$= \quad a{>}b\ \Rightarrow\ 1{+}(div\ a\ 10) \le a \qquad a{>}b \Rightarrow a{>}0$$
$$= \quad \top$$

Proof of last case:

$$a{\ne}b \land a{\le}b \land (b{:=}\ div\ b\ 10.\ \ t{:=}\ t{+}1.\ \ t' \le t{+}a{+}b)\ \ \Rightarrow\ \ t' \le t{+}a{+}b$$
simplify and substitution law twice
$$= \quad a{<}b\ \land\ t' \le t{+}1{+}a{+}(div\ b\ 10)\ \ \Rightarrow\ \ t' \le t{+}a{+}b \qquad \text{connection}$$
$$= \quad a{<}b\ \Rightarrow\ 1{+}(div\ b\ 10) \le b \qquad a{<}b \Rightarrow b{>}0$$
$$= \quad \top$$

Note: The position of a node in an n-ary tree can be given by the path from the root to it, expressed as a sequence of digits base n. The lcp of two node positions is the position of their nearest common ancestor.