290 (factorial space) We can compute $x:= n!$ (factorial) as follows.

$$x:= n! \Leftarrow \textbf{if } n=0 \textbf{ then } x:= 1 \textbf{ else } n:= n-1.\ x:= n!.\ n:= n+1.\ x:= x\times n \textbf{ fi}$$

Each call $x:= n!$ pushes a return address onto a stack, and each return pops an address from the stack. Add a space variable s and a maximum space variable m, with appropriate assignments to them in the program. Find and prove an upper bound on the maximum space used.

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$$\begin{array}{l} m\ge s \Rightarrow m' = m\uparrow(s+n) \Leftarrow \\ \quad \textbf{if } n=0 \textbf{ then } x:= 1 \\ \quad \textbf{else } n:= n-1. \\ \quad\quad s:= s+1.\ m:= m\uparrow s.\ m\ge s \Rightarrow m' = m\uparrow(s+n).\ s:= s-1. \\ \quad\quad n:= n+1.\ x:= x\times n \textbf{ fi} \end{array}$$

Proof: by cases. First case:

$$\begin{array}{lll} & (n=0 \wedge (x:= 1) \Rightarrow (m\ge s \Rightarrow m' = m\uparrow(s+n)) & \text{portation and expand} \\ = & n=0 \wedge x'=1 \wedge n'=n \wedge s'=s \wedge m'=m \wedge m\ge s \Rightarrow m' = m\uparrow(s+n) & \text{context} \\ = & n=0 \wedge x'=1 \wedge n'=n \wedge s'=s \wedge m'=m \wedge m\ge s \Rightarrow \top & \text{base} \\ = & \top & \end{array}$$

Second case:

$$\begin{array}{lll} & (\quad m\ge s \Rightarrow m' = m\uparrow(s+n) & \\ & \Leftarrow (n\ne 0 \wedge (\ n:= n-1.\ s:= s+1. & \\ & \qquad m:= m\uparrow s.\ m\ge s \Rightarrow m' = m\uparrow(s+n).\ s:= s-1. & \\ & \qquad n:= n+1.\ x:= x\times n)) & \text{portation} \\ = & \quad n\ne 0 \wedge m\ge s & \\ & \wedge (\ n:= n-1.\ s:= s+1.\ m:= m\uparrow s.\ m\ge s \Rightarrow m' = m\uparrow(s+n). & \\ & \quad s:= s-1.\ n:= n+1.\ x:= x\times n) & \\ & \Rightarrow m' = m\uparrow(s+n) & \\ & \multicolumn{2}{r}{\text{three substitutions; expand final assignment and two more substitutions}} \\ = & \quad n\ne 0 \wedge m\ge s & \\ & \wedge (\ m\uparrow(s+1) \ge s+1 \Rightarrow m' = m\uparrow(s+1)\uparrow(s+1+n-1). & \\ & \quad x'= x\times(n+1) \wedge n'=n+1 \wedge s'=s-1 \wedge m'=m) & \\ & \Rightarrow m' = m\uparrow(s+n) & \text{simplify first } \uparrow \text{ to } \top \text{ and } +1-1 \text{ to } 0\,. \\ & \multicolumn{2}{r}{\text{In the context where natural } n \text{ is not } 0\,,\ s+n \ge s+1 \text{ so remove } s+1 \text{ from } \uparrow\,.} \\ = & \quad n\ne 0 \wedge m\ge s & \\ & \wedge (m' = m\uparrow(s+n).\ x'= x\times(n+1) \wedge n'=n+1 \wedge s'=s-1 \wedge m'=m) & \\ & \Rightarrow m' = m\uparrow(s+n) & \text{Eliminate . and then use one-point.} \\ = & \quad n\ne 0 \wedge m\ge s \wedge m' = m\uparrow(s+n) & \\ & \Rightarrow m' = m\uparrow(s+n) & \text{specialize} \\ = & \top & \end{array}$$