3 Here are five statements.

(i) This statement is true.

(ii) This statement is false.

(iii) This statement is either true or false.

(iv) This statement is neither true nor false.

(v) This statement is both true and false.

Which of these statements are

(a) true?

(b) false?

(c) either true or false?

(d) neither true nor false?

(e) both true and false?

After trying the question, scroll down to the solution.

§ Let statement (i) be i. Statement (i) says $i=\top$. We don't yet know if statement (i) is true or false. What we do know is that statement (i) says that statement (i) is true.

$$i=(i=\top)$$

A solution is a value for i that makes the expression $\top$.
Both $\top$ and $\bot$ for i make this expression $\top$. It is an equation with two solutions. Statement (i) could be either true or false.

Let statement (ii) be ii. Statement (ii) says $ii=\bot$. We don't yet know if statement (ii) is true or false. What we do know is that statement (ii) says that statement (ii) is false.

$$ii=(ii=\bot)$$

A solution is a value for ii that makes this expression $\top$.
Neither $\top$ nor $\bot$ for ii makes this expression $\top$. It is an equation with no solution.
Statement (ii) is known as the Liar's Paradox.

Let statement (iii) be iii. Statement (iii) says $iii=\top \vee iii=\bot$. We don't yet know if statement (iii) is true or false. What we do know is that statement (iii) says that statement (iii) is either true or false.

$$iii=(iii=\top \vee iii=\bot)$$

A solution is a value for iii that makes this expression $\top$.
Only $\top$ for iii makes this expression $\top$. It is an equation with one solution: $\top$.

Let statement (iv) be iv. Statement (iv) says $iv\neq\top \wedge iv\neq\bot$. We don't yet know if statement (iv) is true or false. What we do know is that statement (iv) says that statement (iv) is neither true nor false.

$$iv=(iv\neq\top \wedge iv\neq\bot)$$

A solution is a value for iv that makes this expression $\top$.
Only $\bot$ for iv makes this expression $\top$. It is an equation with one solution: $\bot$.

Let statement (v) be v. Statement (v) says $v=\top \wedge v=\bot$. We don't yet know if statement (v) is true or false. What we do know is that statement (v) says that statement (v) is both true and false.

$$v=(v=\top \wedge v=\bot)$$

A solution is a value for v that makes this expression $\top$.
Only $\bot$ for v makes this expression $\top$. It is an equation with one solution: $\bot$.

Now we can answer the questions.

(a) Statement (iii) is true.
(b) Statements (iv) and (v) are false.
(c) Statement (i) is either true or false.
(d) Statement (ii) is neither true nor false.
(e) Statement (ii) is both true and false.

Here is a little calculation. The first line says that x is neither true nor false.

$$\begin{aligned} & x\neq\top \wedge x\neq\bot \\ = \quad & x=\bot \wedge x=\top \\ = \quad & x=\top \wedge x=\bot \end{aligned}$$

The last line says that x is both true and false. Saying that x is neither true nor false is equivalent to saying that x is both true and false. And in fact, all three of these lines are equivalent (can be simplified) to just $\bot$.