318 Let $n$ be a natural constant, and let $f$ and $i$ be natural state variables. Define

$$n! \;=\; \Pi i\colon 0,..n\cdot\ i+1 \;=\; 1\times2\times3\times...\times n$$

Prove

$$f'=n! \;\Leftarrow\; f:= 1.\ \ i:= 0.\ \ \textbf{while } i<n \textbf{ do } i:= i+1.\ \ f:= f\times i \textbf{ od}$$

After trying the question, scroll down to the solution.

§ We prove two refinements, namely

$$f'=n! \Leftarrow f:=1.\ i:=0.\ i\le n \Rightarrow f'=f\times n!/i!$$

$$i\le n \Rightarrow f'=f\times n!/i! \Leftarrow \textbf{if } i<n \textbf{ then } i:=i+1.\ f:=f\times i.\ i\le n \Rightarrow f'=f\times n!/i! \textbf{ else } ok \textbf{ fi}$$

First refinement, starting with the right side,

| | | |
|---|---|---|
| | $f:=1.\ i:=0.\ i\le n \Rightarrow f'=f\times n!/i!$ | substitution law twice |
| $=$ | $0\le n \Rightarrow f'=1\times n!/0!$ | simplify |
| $=$ | $f'=n!$ | |

Last refinement, by cases; first case:

| | | |
|---|---|---|
| | $i<n \wedge (i:=i+1.\ f:=f\times i.\ i\le n \Rightarrow f'=f\times n!/i!)$ | substitution law twice |
| $=$ | $i<n \wedge (i+1\le n \Rightarrow f'=f\times(i+1)\times n!/(i+1)!)$ | discharge and simplify |
| $=$ | $i<n \wedge f'=f\times n!/i!$ | specialize |
| $\Rightarrow$ | $f'=f\times n!/i!$ | |
| $\Rightarrow$ | $i\le n \Rightarrow f'=f\times n!/i!$ | |

Last refinement, last case:

| | | |
|---|---|---|
| | $(i\le n \Rightarrow f'=f\times n!/i! \Leftarrow i\ge n \wedge ok)$ | portation |
| $=$ | $i\ge n \wedge ok \wedge i\le n \Rightarrow f'=f\times n!/i!$ | simplify and expand *ok* |
| $=$ | $i=n \wedge f'=f \wedge i'=i \Rightarrow f'=f\times n!/i!$ | context |
| $=$ | $i=n \wedge f'=f \wedge i'=i \Rightarrow \top$ | base |
| $=$ | $\top$ | |