330 (majority vote) The problem is to find, in a given list, the majority item (the item that occurs in more than half the places) if there is one. Letting L be the list and m be a variable whose final value is the majority item, prove that the following program solves the problem.

(a)

```
var e: nat := 0·
for i:= 0;..#L
do  if m = L i then e:= e+1
    else  if i = 2×e then m:= L i.  e:= e+1
          else ok fi fi od
```

(b)

```
var s: nat := 0·
for i:= 0;..#L
do  if m = L i then ok
    else  if i = 2×s then m:= L i
          else s:= s+1 fi fi od
```

After trying the question, scroll down to the solution.

(a) **var** e: $nat := 0\cdot$
for $i := 0;..\#L$
do **if** $m = L\,i$ **then** $e := e+1$
else **if** $i = 2\times e$ **then** $m := L\,i.\ \ e := e+1$
else ok **fi** **od**

§ This is a hard question. Let $N\,x\,i \ = \ ¢\S j{:}\ 0,..i\cdot\ L\,j = x$. So $N\,x\,i$ is the number of occurrences of item x in list L before index i . The problem can be stated formally as

$$\forall x\cdot\ N\,x\,(\#L) > \#L/2 \Rightarrow m' = x$$

or

$$\forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L/2$$

The invariant we need for the **for**-loop is defined as

$$A\,i \ = \ i \le 2\times e \ \wedge\ N\,m\,i \le e \ \wedge \forall x\cdot\ x \neq m \Rightarrow N\,x\,i \le i - e$$

We must prove two refinements:

$$\forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L/2 \ \Leftarrow\ \textbf{var}\ e{:}\ nat := 0\cdot\ \ A\,0 \Rightarrow A'(\#L)$$

$$i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1) \ \Leftarrow$$
if $m = L\,i$ **then** $e := e+1$
else **if** $i = 2\times e$ **then** $m := L\,i.\ \ e := e+1$
else ok **fi**

The first refinement is proven as follows:

var e: $nat := 0\cdot\ \ A\,0 \Rightarrow A'(\#L)$

$= \quad \exists e{:}\ 0\cdot\ \exists e'{:}\ nat\cdot \qquad (0 \le 2\times e \ \wedge\ N\,m\,0 \le e \ \wedge \forall x\cdot\ x \neq m \Rightarrow N\,x\,0 \le 0 - e)$
$\qquad\qquad \Rightarrow\ (\#L \le 2\times e' \ \wedge\ N\,m'(\#L) \le e' \ \wedge \forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L - e')$

$= \quad \exists e'\cdot \qquad (0 \le 0 \ \wedge\ N\,m\,0 \le 0 \ \wedge \forall x\cdot\ x \neq m \Rightarrow N\,x\,0 \le 0)$
$\qquad \Rightarrow\ (\#L \le 2\times e' \ \wedge\ N\,m'(\#L) \le e' \ \wedge \forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L - e')$

the antecedent reduces to $\top$.

In the consequent, drop the second conjunct and rewrite the first

$\Longrightarrow\ \exists e'\cdot\ \ \#L - e' \le \#L/2 \ \wedge\ \forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L - e'$

now use the first conjunct to weaken the last

$\Longrightarrow\ \forall x\cdot\ x \neq m' \Rightarrow N\,x\,(\#L) \le \#L/2$

The second refinement to be proven can be broken into three cases. The first case is

$(i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1) \ \Leftarrow\ m = L\,i \ \wedge\ (e := e+1))$ portation

$= \quad m = L\,i \ \wedge\ (e := e+1) \ \wedge\ i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1)$

$= \qquad m = L\,i \ \wedge\ e' = e+1 \ \wedge\ m' = m \ \wedge\ i{:}\ 0,..\#L$
$\qquad \wedge \quad i \le 2\times e \ \wedge\ N\,m\,i \le e \ \wedge\ (\forall x\cdot\ x \neq m \Rightarrow N\,x\,i \le i - e)$
$\qquad \Rightarrow \quad i+1 \le 2\times e' \ \wedge\ N\,m'(i+1) \le e' \ \wedge\ (\forall x\cdot\ x \neq m' \Rightarrow N\,x\,(i+1) \le i+1-e')$

in the antecedent we have $m' = m = L\,i$, so $N\,m'(i+1) = N\,m\,i + 1$
and for all other x , $N\,x\,i = N\,x\,(i+1)$

$= \quad \top$

The second case is

$(i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1) \ \Leftarrow\ m \neq L\,i \ \wedge\ i = 2\times e \ \wedge\ (m := L\,i.\ \ e := e+1))$ portation

$= \quad m \neq L\,i \ \wedge\ i = 2\times e \ \wedge\ (m := L\,i.\ \ e := e+1) \ \wedge\ i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1)$

$= \qquad m \neq L\,i \ \wedge\ i = 2\times e \ \wedge\ m' = L\,i \ \wedge\ e' = e+1 \ \wedge\ i{:}\ 0,..\#L$
$\qquad \wedge \quad i \le 2\times e \ \wedge\ N\,m\,i \le e \ \wedge\ (\forall x\cdot\ x \neq m \Rightarrow N\,x\,i \le i - e)$
$\qquad \Rightarrow \quad i+1 \le 2\times e' \ \wedge\ N\,m'(i+1) \le e' \ \wedge\ (\forall x\cdot\ x \neq m' \Rightarrow N\,x\,(i+1) \le i+1-e')$

simplify antecedent, and use its equations

$= \qquad m \neq L\,i = m' \ \wedge\ i = 2\times e \ \wedge\ e' = e+1 \ \wedge\ i{:}\ 0,..\#L \ \wedge\ (\forall x\cdot\ N\,x\,i \le e)$
$\qquad \Rightarrow \quad 2\times e+1 \le 2\times(e+1) \ \wedge\ N(L\,i)(i+1) \le e+1 \ \wedge\ (\forall x\cdot\ x \neq L\,i \Rightarrow N\,x\,(i+1) \le e)$

$= \quad \top$

The third case is

$(i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1) \ \Leftarrow\ m \neq L\,i \ \wedge\ i \neq 2\times e \ \wedge\ ok)$ portation

$= \quad m \neq L\,i \ \wedge\ i \neq 2\times e \ \wedge\ ok \ \wedge\ i{:}\ 0,..\#L \ \wedge\ A\,i \ \Rightarrow\ A'(i+1)$

$= \qquad m \neq L\,i \ \wedge\ i \neq 2\times e \ \wedge\ e' = e \ \wedge\ m' = m \ \wedge\ i{:}\ 0,..\#L$

$\wedge \quad i \le 2\times e \wedge N\ m\ i \le e \wedge (\forall x\cdot\ x \ne m \Rightarrow N\ x\ i \le i-e)$

$\Rightarrow \quad i+1 \le 2\times e' \wedge N\ m'(i+1) \le e' \wedge (\forall x\cdot\ x \ne m' \Rightarrow N\ x\ (i+1) \le i+1-e')$

in the antecedent we have $m' = m \ne L\ i$, so $N\ m'(i+1) = N\ m\ i$.

And for all x , $N\ x\ (i+1)$: $N\ x\ i, N\ x\ i + 1$.

$= \quad \top$

Note that the program is correct even though the initial value of variable m is arbitrary.

(b)

```
var s: nat := 0·
for i:= 0;..#L
do  if m = L i then ok
    else if i = 2×s then m:= L i
         else s:= s+1 fi od
```

§ This is the same as part (a) with $s = i-e$.