348 A coin is flipped repeatedly. At each head, natural variable x is decreased by 1 ; at each tail, x is left unchanged. How many flips are there until $x=0$?

After trying the question, scroll down to the solution.

§ The program is

$$x'=0 \Leftarrow \textbf{if } x=0 \textbf{ then } ok \textbf{ else } x:= x - rand\ 2.\ \ t:= t+1.\ \ x'=0 \textbf{ fi}$$

and it is easily proven. We want the probability distribution of t' . After considerable thought, we might propose that it is P , defined as

$$P = \textbf{if } x=0 \textbf{ then } t'=t \textbf{ else } (t' \ge t+x) \times C\ (t'-t-1)\ (x-1) \times 2^{t-t'} \textbf{ fi}$$

where $C\ n\ k$ is the number of ways of choosing k things from n things

$$C\ n\ k = n! / (k! \times (n-k)!) \quad \text{for } 0 \le k \le n$$

where ! is factorial. Before we tackle the main proof, we prove a lemma:

For $0 \le k < n$,

$C\ n\ k + C\ n\ (k+1)$ — definition of C

$= n! / (k! \times (n-k)!) + n! / ((k+1)! \times (n-k-1)!)$ — make common denominator

$= (n! \times (k+1)) / ((k+1)! \times (n-k)!) + (n! \times (n-k)) / ((k+1)! \times (n-k)!)$ — add and factor

$= (n! \times (k+1+n-k)) / ((k+1)! \times (n-k)!)$

$= (n+1)! / ((k+1)! \times (n-k)!)$

$= C\ (n+1)\ (k+1)$

Now for the main proof, we prove

$$P = \textbf{if } x=0 \textbf{ then } t'=t \textbf{ else } x:= x - rand\ 2.\ \ t:= t+1.\ \ P \textbf{ fi}$$

If $x=0$ the equation holds, so assume $x>0$.

$x:= x - rand\ 2.\ \ t:= t+1.\ \ P$

$= (t:= t+1.\ \ P)/2 + (x:= x-1.\ \ t:= t+1.\ \ P)/2$ — replace P and perform substitutions

$= \quad \textbf{if } x=0 \textbf{ then } t'=t+1 \textbf{ else } (t' \ge t+1+x) \times C\ (t'-t-2)\ (x-1) \times 2^{t+1-t'} \textbf{ fi}/2$

$\quad + \textbf{ if } x-1=0 \textbf{ then } t'=t+1 \textbf{ else } (t' \ge t+x) \times C\ (t'-t-2)\ (x-2) \times 2^{t+1-t'} \textbf{ fi}/2$

The first line reduces to the **else**-part because $x>0$.
Division by 2 subtracts 1 from the exponent.

$= \quad (t' \ge t+1+x) \times C\ (t'-t-2)\ (x-1) \times 2^{t-t'}$

$\quad + \textbf{ if } x=1 \textbf{ then } (t'=t+1)/2 \textbf{ else } (t' \ge t+x) \times C\ (t'-t-2)\ (x-2) \times 2^{t-t'} \textbf{ fi}$

SOMEHOW, MAYBE BY CASE ANALYSIS WITH 4 CASES

$= (t' \ge t+x) \times (C\ (t'-t-2)\ (x-1) + C\ (t'-t-2)\ (x-2)) \times 2^{t-t'}$ — use the lemma

$= (t' \ge t+x) \times C\ (t'-t-1)\ (x-1) \times 2^{t-t'}$ — when $x>0$

$= P$