358 (amazing average) Consider the following innocent-looking program, where p is a positive natural variable.

$$loop \;=\; \textbf{if } rand\ 2 \textbf{ then } p:= 2\times p.\ \ t:= t+1.\ \ loop \textbf{ else } ok \textbf{ fi}$$

We repeatedly flip a coin; each time we see a head, we double p, stopping the first time we see a tail.

(a) What is the *loop* distribution?

(b) What is the average final value of t ?

(c) What is the average final value of p ?

After trying the question, scroll down to the solution.

(a) What is the *loop* distribution?

§ $(t' \ge t) \times (p' = 2^{t'-t} \times p) / 2^{t'-t+1}$

Here's the proof.

$$\textbf{if } 1/2 \textbf{ then } p := 2\times p.\ t := t+1.\ (t' \ge t) \times (p' = 2^{t'-t} \times p) / 2^{t'-t+1} \textbf{ else } ok \textbf{ fi}$$

$$= \quad (t' \ge t+1) \times (p' = 2\times 2^{t'-t-1} \times p) / 2^{t'-t-1+1} / 2 \quad + \quad (t'=t) \times (p'=p) / 2$$

$$= \quad (t' \ge t) \times (p' = 2^{t'-t} \times p) / 2^{t'-t+1}$$

(b) What is the average final value of t ?

§ The average value of t' is

$$(t' \ge t) \times (p' = 2^{t'-t} \times p) / 2^{t'-t+1}.\ t \qquad \text{definition of } .$$

$$= \quad \Sigma p'', t''\cdot\ (t'' \ge t) \times (p'' = 2^{t''-t} \times p) / 2^{t''-t+1} \times t'' \qquad \text{sum}$$

$$= \quad t+1$$

On average, the loop body is executed once.

(c) What is the average final value of p ?

§ The average value of p' is

$$(t' \ge t) \times (p' = 2^{t'-t} \times p) / 2^{t'-t+1}.\ p \qquad \text{definition of } .$$

$$= \quad \Sigma p'', t''\cdot\ (t'' \ge t) \times (p'' = 2^{t''-t} \times p) / 2^{t''-t+1} \times p'' \qquad \text{sum}$$

$$= \quad \infty$$

On average, the final value of p is ∞ .