37 (circular numbers) Redesign the axioms for the extended number system to make it circular, so that $+\infty = -\infty$. Be careful with the transitivity of $<$.

After trying the question, scroll down to the solution.

§ I add axiom

$\infty = -\infty$

Now I show that there's an inconsistency by proving $\bot$.

	$\top$	direction
$=$	$-\infty < 0 < 1 < \infty$	transitivity
$\Rightarrow$	$-\infty < \infty$	use the new axiom
$=$	$\infty < \infty$	double negation
$=$	$\neg\neg(\infty<\infty)$	irreflexivity
$=$	$\neg\top$	evaluation rule
$=$	$\bot$	

Whenever there's an inconsistency, we have to weaken or withdraw one or more of the axioms used in the proof of inconsistency, so that the proof can no longer be made. The question asks us to keep the new axiom. I propose that we begin by weakening the direction law to just

$0 < 1$

and not say how $-\infty$ and ∞ compare with finite numbers and with each other. That saves us from the above proof, but now we need to make other changes. We need to delete the extremes law

$-\infty \le x \le \infty$

And in any law having the antecedent that compares ∞ or $-\infty$, for example $-\infty<x<\infty$ or $x<\infty$ or $-\infty<x$, change the antecedent to $x \ne \infty$. The two absorption laws

$-\infty<x \;\Rightarrow\; \infty+x = \infty$

$x<\infty \;\Rightarrow\; -\infty+x = -\infty$

become one law

$x \ne \infty \;\Rightarrow\; \infty+x = \infty$

Likewise the two absorption laws

$x<\infty \;\Rightarrow\; \infty-x = -\infty$

$-\infty<x \;\Rightarrow\; -\infty - x = -\infty$

become one law

$x \ne \infty \;\Rightarrow\; \infty-x = \infty$

And the two absorption laws

$0<x \;\Rightarrow\; x\times\infty = \infty$

$0<x \;\Rightarrow\; x \times -\infty \;=\; -\infty$

become one law

$x \ne 0 \;\Rightarrow\; x\times\infty = \infty$

I think consistency is now restored (I hope). And now we have an opportunity to add a new axiom

$x \ne 0 \;\Rightarrow\; x/0 = \infty$

We couldn't have it before because $0 = -0$ so we would have

$\infty = 1/0 = 1/(-0) = -(1/0) = -\infty$

which was inconsistent. But now it's ok.