378 Let R be a relation of naturals $R: nat \to nat \to bin$ that is monotonic in its second parameter

$$\forall i, j \cdot R\ i\ j \Rightarrow R\ i\ (j+1)$$

Prove

$$\exists i \cdot \forall j \cdot R\ i\ j \ = \ \forall j \cdot \exists i \cdot R\ i\ j\ .$$

After trying the question, scroll down to the solution.

§ One of the versions of induction is

$$\forall n\text{: } nat\cdot\ Pn \Rightarrow P(n+1) \quad \Longrightarrow \quad \forall n\text{: } nat\cdot\ P\,0 \Rightarrow P\,n$$

If we change the variable to j and $P\,n$ to $R\,i\,j$, the left side of this induction becomes the given information. So starting with the given information

$\forall i, j\cdot\ R\,i\,j \Rightarrow R\,i\,(j+1)$ — specialize local i to nonlocal i

$\Longrightarrow\ \forall j\cdot\ R\,i\,j \Rightarrow R\,i\,(j+1)$ — use induction

$\Longrightarrow\ \forall j\cdot\ R\,i\,0 \Rightarrow R\,i\,j$ — push $\forall j$ into the consequent

$=\ R\,i\,0 \Rightarrow \forall j\cdot R\,i\,j$

The last line is a lemma we'll need in a moment. Now

$\forall j\cdot\ \exists i\cdot\ R\,i\,j$ — use specialization

$\Longrightarrow\ \exists i\cdot\ R\,i\,0$ — use lemma

$\Longrightarrow\ \exists i\cdot\ \forall j\cdot\ R\,i\,j$

which is one direction of the theorem. The other direction is the semicommutative law.