383 The Fibonacci numbers *fib* n are defined as follows.

$$fib\ 0 = 0$$
$$fib\ 1 = 1$$
$$fib\ (n+2) = fib\ n + fib\ (n+1)$$

Prove

After trying the question, scroll down to the solution.

§ I need a kind of induction that allows us to assume two previous cases as induction hypothesis in order to prove the next case. So I begin by proving the necessary induction law. All quantifications are over *nat* .

$$\begin{array}{lll}
& \top & \text{predicate form of } nat \text{ induction}\\
= & P\,0 \land (\forall n\cdot P\,n \Rightarrow P(n+1)) \;\Rightarrow\; (\forall n\cdot P\,n) & \text{let } P\,n = Q\,n \land Q(n+1)\\
\Longrightarrow & Q\,0 \land Q\,1 \land (\forall n\cdot Q\,n \land Q(n+1) \Rightarrow Q(n+1) \land Q(n+2)) \;\Rightarrow\; (\forall n\cdot Q\,n \land Q(n+1)) & \\
= & Q\,0 \land Q\,1 \land (\forall n\cdot Q\,n \land Q(n+1) \Rightarrow Q(n+2)) \;\Rightarrow\; (\forall n\cdot Q\,n \land Q(n+1)) & \\
\Longrightarrow & Q\,0 \land Q\,1 \land (\forall n\cdot Q\,n \land Q(n+1) \Rightarrow Q(n+2)) \;\Rightarrow\; (\forall n\cdot Q\,n) &
\end{array}$$

(a) $fib\ (gcd\ n\ m) \;=\; gcd\ (fib\ n)\ (fib\ m)$

where *gcd* is the greatest common divisor.

(b) $fib\ n \times fib\ (n+2) \;=\; (fib\ (n+1))^2 \;-\; (-1)^n$

(c) $fib\ (n+m+1) \;=\; fib\ n \times fib\ m \;+\; fib\ (n+1) \times fib\ (m+1)$

§ By induction on n . First, $n=0$.

$$\begin{array}{ll}
& fib\ n \times fib\ m \;+\; fib\ (n+1) \times fib\ (m+1)\\
= & fib\ 0 \times fib\ m \;+\; fib\ 1 \times fib\ (m+1)\\
= & 0 \times fib\ m \;+\; 1 \times fib\ (m+1)\\
= & fib\ (0+m+1)\\
= & fib\ (n+m+1)
\end{array}$$

This induction needs a second base case: $n=1$.

$$\begin{array}{ll}
& fib\ n \times fib\ m \;+\; fib\ (n+1) \times fib\ (m+1)\\
= & fib\ 1 \times fib\ m \;+\; fib\ 2 \times fib\ (m+1)\\
= & 1 \times fib\ m \;+\; 1 \times fib\ (m+1)\\
= & fib\ (m+2)\\
= & fib\ (n+m+1)
\end{array}$$

Now we are entitled to assume two previous cases:

$$\begin{array}{lll}
fib\ (n+m-1) & = & fib\ (n-2) \times fib\ m \;+\; fib\ (n-1) \times fib\ (m+1)\\
fib\ (n+m) & = & fib\ (n-1) \times fib\ m \;+\; fib\ n \times fib\ (m+1)
\end{array}$$

Add the left sides, and add the right sides.

$$\begin{array}{ll}
& fib\ (n+m-1) + fib\ (n+m)\\
= & (fib\ (n-2) + fib\ (n-1)) \times fib\ m \;+\; (fib\ (n-1) + fib\ n) \times fib\ (m+1)
\end{array}$$

Use the definition of *fib* three times to obtain the desired result.

(d) $fib\ (n+m+2) \;=\; fib\ n \times fib\ (m+1) \;+\; fib\ (n+1) \times fib\ m \;+\; fib\ (n+1) \times fib\ (m+1)$

§ This proof is similar to part (c). First, $n=0$.

$$fib\ (m+2) \;=\; 0 \times fib\ (m+1) \;+\; 1 \times fib\ m \;+\; 1 \times fib\ (m+1)$$

which follows from the definition of *fib* . Now the second base case: $n=1$.

$$fib\ (m+3) \;=\; 1 \times fib\ (m+1) \;+\; 1 \times fib\ m \;+\; 1 \times fib\ (m+1)$$

add the last two terms

$$= \quad fib\ (m+3) \;=\; fib\ (m+1) \;+\; fib\ (m+2)$$

which again follows. Now we are entitled to assume two previous cases:

$$\begin{array}{lll}
fib\ (n+m) & = & fib\ (n-2) \times fib\ (m+1) \;+\; fib\ (n-1) \times fib\ m \;+\; fib\ (n-1) \times fib\ (m+1)\\
fib\ (n+m+1) & = & fib\ (n-1) \times fib\ (m+1) \;+\; fib\ n \times fib\ m \;+\; fib\ n \times fib\ (m+1)
\end{array}$$

Add the left sides, and add the right sides, using the definition on corresponding terms, to get the desired result.

(e) $fib\ (2\times n+1) \;=\; (fib\ n)^2 \;+\; (fib\ (n+1))^2$

§ This follows from part (c): just take $m=n$.

(f) $fib\ (2\times n+2) \;=\; 2 \times fib\ n \times fib\ (n+1) \;+\; (fib\ (n+1))^2$

§ This follows from part (d): just take $m=n$.

(g) $\forall n, k: nat\cdot\ fib\ (k\times n) : nat \times fib\ n$