390 Let $A \backslash B$ be the difference between bunch A and bunch B. The operator $\backslash$ has precedence level 4, and is defined by the axiom

$$x\colon A \backslash B \;=\; x\colon A \;\wedge\; \neg\, x\colon B$$

For each of the following fixed-point equations, what does recursive construction yield? Does it satisfy the fixed-point equation?

(a) $Q \;=\; nat \backslash (Q{+}3)$

§

$$Q_0 \;=\; null$$
$$Q_1 \;=\; nat \backslash (null{+}3) \;=\; nat \backslash null \;=\; nat$$
$$Q_2 \;=\; nat \backslash (nat{+}3) \;=\; 0, 1, 2$$
$$Q_3 \;=\; nat \backslash ((0, 1, 2){+}3) \;=\; nat \backslash (3, 4, 5) \;=\; 0, 1, 2, nat{+}6$$
$$Q_4 \;=\; nat \backslash ((0, 1, 2, nat{+}6){+}3) \;=\; nat \backslash (3, 4, 5, nat{+}9) \;=\; 0, 1, 2, 6, 7, 8$$
$$Q_5 \;=\; nat \backslash ((0, 1, 2, 6, 7, 8){+}3) \;=\; nat \backslash (3, 4, 5, 9, 10, 11)$$
$$=\; 0, 1, 2, 6, 7, 8, nat{+}12$$

Time for a guess. It looks like there are two patterns: the even index pattern and the odd index pattern. So I guess

$$Q_{2\times n} \;=\; 6\times(0,..n) + (0,..3)$$
$$Q_{2\times n+1} \;=\; 6\times(0,..n) + (0,..3),\; (6\times n,..\infty)$$

From the even case, I propose

$$Q_\infty \;=\; 6\times nat + (0,..3)$$

and now I have to check whether it satisfies the equation. Starting with the right side,

$$nat \backslash (Q_\infty{+}3)$$
$$=\; nat \backslash (6\times nat + (0,..3) + 3)$$
$$=\; nat \backslash (6\times nat + (3,..6)) \qquad \text{here is an informal expansion}$$
$$=\; nat \backslash ((0, 6, 12, 18, 24, ...) + (3,..6)) \qquad \text{and an informal addition}$$
$$=\; nat \backslash (3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23, 27, 28, 29, ...)$$
$$=\; 0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, ...$$
$$=\; (0, 6, 12, 18, 24, ...) + (0,..3)$$
$$=\; 6\times nat + (0,..3)$$
$$=\; Q_\infty$$

So it does satisfy the equation. From the odd case, we can't make a proposal because we can't simplify $\infty,..\infty$.

(b) $D \;=\; 0, (D{+}1) \backslash (D{-}1)$

§

$$D_0 \;=\; null$$
$$D_1 \;=\; 0, (D_0{+}1) \backslash (D_0{-}1)$$
$$= 0, (null{+}1) \backslash (null{-}1)$$
$$= 0, null \backslash null$$
$$= 0$$
$$D_2 \;=\; 0, (D_1{+}1) \backslash (D_1{-}1)$$
$$=\; 0, (0{+}1) \backslash (0{-}1)$$
$$=\; 0, 1 \backslash {-1}$$
$$=\; 0, 1$$
$$D_3 \;=\; 0, (D_2{+}1) \backslash (D_2{-}1)$$
$$=\; 0, ((0, 1){+}1) \backslash ((0, 1){-}1)$$
$$=\; 0, (1, 2) \backslash (-1, 0)$$
$$=\; 0, 1, 2$$
$$D_4 \;=\; 0, (D_3{+}1) \backslash (D_3{-}1)$$
$$=\; 0, ((0, 1, 2){+}1) \backslash ((0, 1, 2){-}1)$$
$$=\; 0, (1, 2, 3) \backslash (-1, 0, 1)$$
$$=\; 0, 2, 3$$

$$\begin{aligned} D_5 &= 0, (D_4+1) \setminus (D_4-1) \\ &= 0, ((0, 2, 3)+1) \setminus ((0, 2, 3)-1) \\ &= 0, (1, 3, 4) \setminus (-1, 1, 2) \\ &= 0, 3, 4 \\ D_6 &= 0, (D_5+1) \setminus (D_5-1) \\ &= 0, ((0, 3, 4)+1) \setminus ((0, 3, 4)-1) \\ &= 0, (1, 4, 5) \setminus (-1, 2, 3) \\ &= 0, 1, 4, 5 \\ D_7 &= 0, (D_6+1) \setminus (D_6-1) \\ &= 0, ((0, 1, 4, 5)+1) \setminus ((0, 1, 4, 5)-1) \\ &= 0, (1, 2, 5, 6) \setminus (-1, 0, 3, 4) \\ &= 0, 1, 2, 5, 6 \\ D_8 &= 0, (D_7+1) \setminus (D_7-1) \\ &= 0, ((0, 1, 2, 5, 6)+1) \setminus ((0, 1, 2, 5, 6)-1) \\ &= 0, (1, 2, 3, 6, 7) \setminus (-1, 0, 1, 4, 5) \\ &= 0, 2, 3, 6, 7 \end{aligned}$$

It's still hard to see the patterns, so maybe we have to go a bit farther. Then we see

$$\begin{aligned} D_{4\times n+1} &= 0, 4\times(0,..n) + (3, 4) \\ D_{4\times n+2} &= 0, 1, 4\times(0,..n) + (4, 5) \\ D_{4\times n+3} &= 0, 1, 2, 4\times(0,..n) + (5, 6) \\ D_{4\times n+4} &= 0, 2, 3, 4\times(0,..n) + (6, 7) \end{aligned}$$

We have a choice of four possible answers for D_∞, but none of them satisfies the equation. Recursive construction fails.

(c) $E = nat \setminus (E+1)$

§

$$\begin{aligned} E_0 &= null \\ E_1 &= nat \\ E_2 &= 0 \\ E_3 &= 0, nat+2 \\ E_4 &= 0, 2 \\ E_5 &= 0, 2, nat+4 \\ E_{2\times n} &= 2\times(0,..n) \\ E_{2\times n+1} &= 2\times(0,..n), nat+2\times n \end{aligned}$$

From the even case, we propose

$E_\infty = 2\times nat$

which satisfies the equation. From the odd case, we propose

$E_\infty = 2\times nat, \infty$

which does not satisfy the equation.

(d) $F = 0, (nat \setminus F)+1$

§

$$\begin{aligned} F_0 &= null \\ F_1 &= nat \\ F_2 &= 0 \\ F_3 &= 0, nat+2 \\ F_4 &= 0, 2 \\ F_5 &= 0, 2, nat+4 \\ F_{2\times n} &= 2\times(0,..n) \\ F_{2\times n+1} &= 2\times(0,..n), nat+2\times n \end{aligned}$$

From the even case, we propose

$F_\infty = 2\times nat$

which satisfies the fixed-point equation. From the odd case, we propose

$$F_\infty = 2 \times nat, \infty$$

which does not satisfy the fixed-point equation.