407 Let all variables be integer. Add recursive time. Using recursive construction, find a fixed-point of

(a) $skip = \textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ skip.\ i{:=}\ i+1 \textbf{ else } ok \textbf{ fi}$

§ Adding recursive time,

$$skip = \textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ t{:=}\ t+1.\ skip.\ i{:=}\ i+1 \textbf{ else } ok \textbf{ fi}$$
$$skip_0 = t' \ge t$$
$$skip_{n+1} = \textbf{if } i\ge n \textbf{ then } t' \ge t+n+1 \textbf{ else if } 0\le i<n \textbf{ then } t{:=}\ t+i+1 \textbf{ else } ok \textbf{ fi fi}$$
$$skip_\infty = \textbf{if } i\ge 0 \textbf{ then } t{:=}\ t+i+1 \textbf{ else } ok \textbf{ fi}$$

To show it's a fixed-point, start with the right side of the definition of $skip$, but substitute $skip_\infty$ in place of $skip$,

$\textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ t{:=}\ t+1.\ \textbf{if } i\ge 0 \textbf{ then } t{:=}\ t+i+1 \textbf{ else } ok \textbf{ fi}.\ i{:=}\ i+1 \textbf{ else } ok \textbf{ fi}$

distribute $i{:=}\ i+1$ into preceding **if**

$= \textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ t{:=}\ t+1.\ \textbf{if } i\ge 0 \textbf{ then } t{:=}\ t+i+1.\ i{:=}\ i+1 \textbf{ else } ok.\ i{:=}\ i+1 \textbf{ fi else } ok \textbf{ fi}$

replace first $i{:=}\ i+1$ and ok is identity for .

$= \textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ t{:=}\ t+1.\ \textbf{if } i\ge 0 \textbf{ then } t{:=}t+i+1.\ i'=i+1 \wedge t'=t \textbf{ else } i{:=}\ i+1 \textbf{ fi else } ok \textbf{ fi}$

substitution law in second **then**-part

$= \textbf{if } i\ge 0 \textbf{ then } i{:=}\ i-1.\ t{:=}\ t+1.\ \textbf{if } i\ge 0 \textbf{ then } i'=i+1 \wedge t'=t+i+1 \textbf{ else } i{:=}\ i+1 \textbf{ fi else } ok \textbf{ fi}$

replace $i{:=}\ i+1$

$= \textbf{if } i\ge 0 \textbf{ then } i{:=}i-1.\ t{:=}\ t+1.\ \textbf{if } i\ge 0 \textbf{ then } i'=i+1 \wedge t'=t+i+1 \textbf{ else } i'=i+1 \wedge t'=t \textbf{ fi else } ok \textbf{ fi}$

substitution law twice more

$= \textbf{if } i\ge 0 \textbf{ then if } i-1\ge 0 \textbf{ then } i'=i-1+1 \wedge t'=t+1+i-1+1 \textbf{ else } i'=i-1+1 \wedge t'=t+1 \textbf{ fi else } ok \textbf{ fi}$

simplify

$= \textbf{if } i\ge 0 \textbf{ then if } i\ge 1 \textbf{ then } i'=i \wedge t'=t+i+1 \textbf{ else } i'=i \wedge t'=t+1 \textbf{ fi else } ok \textbf{ fi}$ use := twice

$= \textbf{if } i\ge 0 \textbf{ then if } i\ge 1 \textbf{ then } t{:=}\ t+i+1 \textbf{ else } t{:=}\ t+1 \textbf{ fi else } ok \textbf{ fi}$

In the first **else**-part the context is $i\ge 0 \wedge \neg(i\ge 1)$ which is $i=0$

$= \textbf{if } i\ge 0 \textbf{ then if } i\ge 1 \textbf{ then } t{:=}\ t+i+1 \textbf{ else } t{:=}\ t+i+1 \textbf{ fi else } ok \textbf{ fi}$ case idempotent

$= \textbf{if } i\ge 0 \textbf{ then } t{:=}\ t+i+1 \textbf{ else } ok \textbf{ fi}$

and we get $skip_\infty$ again, so it is a fixed-point.

(b) $inc = ok \vee (i{:=}\ i+1.\ inc)$

§ Adding recursive time,

$$inc = ok \vee (i{:=}\ i+1.\ t{:=}\ t+1.\ inc)$$

Now recursive construction. Starting with $\top$,

$$inc_0 = \top$$
$$inc_1 = ok \vee (i{:=}\ i+1.\ t{:=}\ t+1.\ inc_0)$$
$$= ok \vee \top$$
$$= \top$$

We have converged, and found that $\top$ is a fixed-point. Perhaps we'll get something more interesting if we start with $t'\ge t$.

$$inc_0 = t'\ge t$$
$$inc_1 = ok \vee (i{:=}\ i+1.\ t{:=}\ t+1.\ inc_0)$$
$$= i'=i \wedge t'=t \ \vee\ t'\ge t+1$$
$$inc_2 = ok \vee (i{:=}\ i+1.\ t{:=}\ t+1.\ inc_1)$$
$$= i'=i \wedge t'=t \ \vee\ i'=i+1 \wedge t'=t+1 \ \vee\ t'\ge t+2$$

I'm ready to guess

$$inc_n = (\exists m\colon 0,..n\cdot\ i'=i+m \wedge t'=t+m) \ \vee\ t'\ge t+n$$
$$inc_\infty = (\exists m\colon nat\cdot\ i'=i+m \wedge t'=t+m) \ \vee\ t'=\infty$$

Now I must test inc_∞ to see if it's a fixed-point.

$ok \vee (i{:=}\ i+1.\ t{:=}\ t+1.\ inc_\infty)$

$= i'=i \wedge t'=t \ \vee\ (\exists m\colon nat\cdot\ i'=i+1+m \wedge t'=t+1+m) \ \vee\ t'=\infty$

$= (\exists m\colon nat\cdot\ i'=i+m \wedge t'=t+m) \ \vee\ t'=\infty$

$= \quad inc_\infty$

Starting with $\bot$ we get

$inc_0 \quad = \quad \bot$

$inc_1 \quad = \quad ok \lor (i{:=}\ i{+}1.\ \ t{:=}\ t{+}1.\ \ inc_0)$

$\qquad = \quad i'{=}i \land t'{=}t$

$inc_2 \quad = \quad ok \lor (i{:=}\ i{+}1.\ \ t{:=}\ t{+}1.\ \ inc_1)$

$\qquad = \quad i'{=}i \land t'{=}t \ \ \lor\ \ i'{=}i{+}1 \land t'{=}t{+}1$

$inc_n \quad = \quad (\exists m\text{: } 0,..n\cdot\ i'{=}i{+}m \land t'{=}t{+}m)$

$inc_\infty \quad = \quad (\exists m\text{: } nat\cdot\ i'{=}i{+}m \land t'{=}t{+}m)$

and it is a fixed-point, and it's implementable too!

(c) $\quad sqr \quad = \quad$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ sqr$ **fi**

§ $\quad sqr_0 \quad = \quad t'{\ge}t$

$sqr_1 \quad = \quad$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ sqr_0$ **fi**

$\quad = \quad$ **if** $i{=}0$ **then** ok **else** $t' \ge t{+}1$

$sqr_2 \quad = \quad$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ sqr_1$ **fi**

$\quad = \quad$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$
if $i{=}0$ **then** ok **else** $t' \ge t{+}1$ **fi fi**

$\quad = \quad$ **if** $i{=}0$ **then** ok
else if $i{-}1{=}0$ **then** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1$
else $t' \ge t{+}2$ **fi fi**

$\quad = \quad$ **if** $i{=}0$ **then** $s{:=}\ s{+}0.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}0$
else if $i{=}1$ **then** $s{:=}\ s{+}1.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}1$
else $t' \ge t{+}2$ **fi fi**

$sqr_3 \quad = \quad$ **if** $i{=}0$ **then** ok
else $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$
if $i{=}0$ **then** $s{:=}\ s{+}0.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}0$
else if $i{=}1$ **then** $s{:=}\ s{+}1.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}1$
else $t' \ge t{+}2$ **fi fi fi**

$\quad = \quad$ **if** $i{=}0$ **then** ok
else if $i{=}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$
$s{:=}\ s{+}0.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}0$
else if $i{=}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ \ i{=}\ i{-}1.\ t{:=}\ t{+}1.$
$s{:=}\ s{+}1.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}1$
else $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ t' \ge t{+}2$ **fi fi fi**

$\quad = \quad$ **if** $i{=}0$ **then** $s{:=}\ s{+}0.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}0$
else if $i{=}1$ **then** $s{:=}\ s{+}1.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}1$
else if $i{=}2$ **then** $s{:=}\ s{+}4.\ \ i{:=}\ 0.\ \ t{:=}\ t{+}2$
else $t' \ge t{+}3$ **fi fi fi**

$sqr_n \quad = \quad$ **if** $0{\le}i{<}n$ **then** $s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$ **else** $t' \ge t{+}n$ **fi**

$sqr_\infty \quad = \quad$ **if** $0{\le}i$ **then** $s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$ **else** $t'{=}\infty$ **fi**

Now we test to see if sqr_∞ is a fixed-point.

if $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$
if $0{\le}i$ **then** $s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$ **else** $t'{=}\infty$ **fi fi**

$=$ **if** $i{=}0$ **then** ok
else if $0{\le}i{-}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$
$s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$
else $s{:=}\ s + 2{\times}i - 1.\ \ i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ t'{=}\infty$ **fi fi**

$=$ **if** $i{=}0$ **then** ok
else if $1{\le}i$ **then** $s{:=}\ s + 2{\times}i - 1 + (i{-}1)^2.\ \ t{:=}\ t{+}1{+}i{-}1.\ \ i{:=}\ 0$
else $t'{=}\infty$ **fi fi**

$=$ **if** $i{=}0$ **then** $s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$

else if $1\le i$ **then** $s{:=}\ s{+}i^2.\ \ t{:=}\ t{+}i.\ \ i{:=}\ 0$
else $t'{=}\infty$ **fi fi**

$=\quad sqr_\infty$

(d) $fac \;=\;$ **if** $i{=}0$ **then** $f{:=}\ 1$ **else** $i{:=}\ i{-}1.\ \ fac.\ \ i{:=}\ i{+}1.\ \ f{:=}\ f{\times}i$ **fi**

§ Adding time,

$fac \;=\;$ **if** $i{=}0$ **then** $f{:=}\ 1$ **else** $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ fac.\ \ i{:=}\ i{+}1.\ \ f{:=}\ f{\times}i$ **fi**

Recursive construction starting with $t'{\ge}t$ produces

$fac_n \;=\;$ **if** $0{\le}i{<}n$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{\ge}t{+}n$ **fi**

where $i!$ is " i factorial". Replacing n with ∞ produces

$fac_\infty \;=\;$ **if** $0{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi**

Now we see if fac_∞ is a fixed-point. Starting with the right side of the fac equation,

if $i{=}0$ **then** $f{:=}\ 1$ **else** $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.\ \ fac.\ \ i{:=}\ i{+}1.\ \ f{:=}\ f{\times}i$ **fi** replace fac with fac_∞

$=$ **if** $i{=}0$ **then** $f{:=}\ 1$ — expand assignment
else $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$ **if** $0{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi**. $i{:=}\ i{+}1.\ \ f{:=}\ f{\times}i$ **fi**
combine and expand the final two assignments

$=$ **if** $i{=}0$ **then** $f'{=}1 \wedge i'{=}i \wedge t'{=}t$ — use **if**-context in **then**-part
else $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$ **if** $0{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi**.
$i' = i{+}1 \wedge f' = f{\times}(i{+}1) \wedge t'{=}t$ **fi** — distribute this line into **then** and **else** parts

$=$ **if** $i{=}0$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$
else $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$ **if** $0{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i.\ \ i' = i{+}1 \wedge f' = f{\times}(i{+}1) \wedge t'{=}t$
else $t'{=}\infty.\ \ i' = i{+}1 \wedge f' = f{\times}(i{+}1) \wedge t'{=}t$ **fi fi** — dep't comp.

$=$ **if** $i{=}0$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$
else $i{:=}\ i{-}1.\ \ t{:=}\ t{+}1.$ **if** $0{\le}i$ **then** $f'{=}(i{+}1)! \wedge i'{=}i{+}1 \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi fi**
substitution law twice

$=$ **if** $i{=}0$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$
else if $1{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi fi** — combine $i{=}0$ and $1{\le}i$ cases

$=$ **if** $0{\le}i$ **then** $f'{=}i! \wedge i'{=}i \wedge t'{=}t{+}i$ **else** $t'{=}\infty$ **fi**

Therefore fac_∞ is a fixed-point.

(e) $chs \;=\;$ **if** $a{=}b$ **then** $c{:=}\ 1$ **else** $a{:=}\ a{-}1.\ \ chs.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$ **fi**

§ $chs_0 \;=\; t'{\ge}t$

$chs_1 \;=\;$ **if** $a{=}b$ **then** $c{:=}\ 1$ **else** $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ chs_0.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$ **fi**

At this point we need to know that $c{\times}a/(a{-}b){:}\ int$ and we don't.
But this whole procedure just generates a candidate that needs to be tested.
So we carry on as if $c{\times}a/(a{-}b){:}\ int$

$=$ **if** $a{=}b$ **then** $c{:=}\ 1$ **else** $t' \ge t{+}1$ **fi**

$chs_2 \;=\;$ **if** $a{=}b$ **then** $c{:=}\ 1$ **else** $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ chs_1.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$ **fi**

$=$ **if** $a{=}b$ **then** $c{:=}\ 1$
else $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.$ **if** $a{=}b$ **then** $c{:=}\ 1$ **else** $t' \ge t{+}1$ **fi**.
$a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$ **fi**

$=$ **if** $a{=}b$ **then** $c{:=}\ 1$
else if $a{-}1{=}b$ **then** $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ c{:=}\ 1.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$
else $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ t' \ge t{+}1.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$ **fi fi**

$=$ **if** $a{=}b$ **then** $c{:=}\ 1$
else if $a{-}1{=}b$ **then** $t{:=}\ t{+}1.\ \ c{:=}\ a$
else $t' \ge t{+}2$ **fi fi**

$chs_3 \;=\;$ **if** $a{=}b$ **then** $c{:=}\ 1$
else $a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.$
if $a{=}b$ **then** $c{:=}\ 1$
else if $a{-}1{=}b$ **then** $t{:=}\ t{+}1.\ \ c{:=}\ a$
else $t' \ge t{+}2$ **fi fi**.

$a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else if $a-1=b$ **then** $a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ c{:=}\ 1.\ \ a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$
else if $a-2=b$ **then** $a{:=}\ a-1.\ t{:=}\ t+1.\ t{:=}\ t+1.\ c{:=}\ a.\ a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$
else $a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ t' \ge t+2.\ \ a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi fi fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else if $a-1=b$ **then** $t{:=}\ t+1.\ \ c{:=}\ a$
else if $a-2=b$ **then** $t{:=}\ t+2.\ \ c{:=}\ a\times(a-1)/2$
else $t' \ge t+3$ **fi fi fi**

$chs_4 =$ **if** $a=b$ **then** $c{:=}\ 1$
else $a{:=}\ a-1.\ \ t{:=}\ t+1.$
if $a=b$ **then** $c{:=}\ 1$
else if $a-1=b$ **then** $t{:=}\ t+1.\ \ c{:=}\ a$
else if $a-2=b$ **then** $t{:=}\ t+2.\ \ c{:=}\ a\times(a-1)/2$
else $t' \ge t+3$ **fi fi fi**.
$a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else if $a-1=b$ **then** $t{:=}\ t+1.\ \ c{:=}\ a$
else if $a-2=b$ **then** $t{:=}\ t+2.\ \ c{:=}\ a\times(a-1)/2$
else if $a-3=b$ **then** $t{:=}\ t+3.\ \ c{:=}\ a\times(a-1)\times(a-2)/(2\times 3)$
else $t' \ge t+4$ **fi fi fi fi**

$chs_n =$ **if** $b \le a < b+n$ **then** $t{:=}\ t+a-b.\ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1])$ **else** $t' \ge t+n$ **fi**

$chs_\infty =$ **if** $a\ge b$ **then** $t{:=}\ t+a-b.\ \ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1]$ **else** $t'=\infty$ **fi**

Now I test to see if chs_∞ is a fixed-point.

if $a=b$ **then** $c{:=}\ 1$ **else** $a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ chs_\infty.\ \ a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else $a{:=}\ a-1.\ \ t{:=}\ t+1.$
if $a\ge b$ **then** $t{:=}\ t+a-b.\ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1]$ **else** $t'=\infty$ **fi**.
$a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else if $a-1\ge b$ **then** $a{:=}\ a-1.\ \ t{:=}\ t+1.$
$t{:=}\ t+a-b.\ \ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1].$
$a{:=}\ a+1.\ c{:=}\ c\times a/(a-b)$
else $a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ t'=\infty.\ \ a{:=}\ a+1.\ \ c{:=}\ c\times a/(a-b)$ **fi fi**

$=$ **if** $a=b$ **then** $c{:=}\ 1$
else if $a>b$ **then** $t{:=}\ t+a-b.\ \ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1]$
else $t'=\infty$ **fi fi**

$=$ **if** $a\ge b$ **then** $t{:=}\ t+a-b.\ \ c{:=}\ \Pi[b+1;..a+1]/\Pi[1;..a-b+1]$ **else** $t'=\infty$ **fi**

$=$ chs_∞

So chs_∞ is a fixed-point. Note that for $1\le b\le a$, c' is the number of ways of choosing b things from a things.

(f) foo $=$ **if** $i=0$ **then** $i{:=}\ 3$ **else** foo **fi**

§ $foo_0 = \top$

$foo_1 =$ **if** $i=0$ **then** $i{:=}\ 3$ **else** $t{:=}\ t+1.\ \ \top$ **fi**
$=$ **if** $i=0$ **then** $i'=3 \land t'=t$ **else** $\top$ **fi**
$=$ $i=0 \Rightarrow i'=3 \land t'=t$

$foo_2 =$ **if** $i=0$ **then** $i{:=}\ 3$ **else** $t{:=}\ t+1.\ \ i=0 \Rightarrow i'=3 \land t'=t$ **fi**
$=$ **if** $i=0$ **then** $i'=3 \land t'=t$ **else** $i=0 \Rightarrow i'=3 \land t'=t+1$ **fi** context
$=$ **if** $i=0$ **then** $i'=3 \land t'=t$ **else** $\top$ **fi**
$=$ $i=0 \Rightarrow i'=3 \land t'=t$

$= \ foo_1$

The weakest fixed-point (solution) $i{=}0 \Rightarrow i'{=}3 \wedge t'{=}t$ has been found.

(g) $bar \ = \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i{:=}\ 3\ \textbf{else}\ bar.\ i{:=}\ 3\ \textbf{fi}$

§

$$bar_0 \ = \ \top$$

$$bar_1 \ = \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i{:=}\ 3\ \textbf{else}\ t{:=}\ t{+}1.\ \top.\ i{:=}\ 3\ \textbf{fi}$$

$$= \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ t{:=}\ t{+}1.\ \top.\ i'{=}3 \wedge t'{=}t\ \textbf{fi}$$

$$= \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ t{:=}\ t{+}1.\ (\exists i'', t''\cdot\ \top \wedge i'{=}3 \wedge t'{=}t'')\ \textbf{fi}$$

$$= \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ t{:=}\ t{+}1.\ i'{=}3\ \textbf{fi}$$

$$= \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ i'{=}3\ \textbf{fi}$$

$$= \ \textbf{if}\ i{=}1\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ i'{=}3\ \textbf{fi}$$

$$= \ i'{=}3 \wedge \textbf{if}\ i{=}1\ \textbf{then}\ t'{=}t\ \textbf{else}\ \top\ \textbf{fi}$$

$$= \ i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t)$$

$$bar_2 \ = \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i{:=}\ 3\ \textbf{else}\ t{:=}\ t{+}1.\ i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t).\ i{:=}\ 3\ \textbf{fi}$$

$$= \ i{:=}\ i{-}1.\ \textbf{if}\ i{=}0\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t{+}1)\ \textbf{fi}$$

$$= \ \textbf{if}\ i{=}1\ \textbf{then}\ i'{=}3 \wedge t'{=}t\ \textbf{else}\ i'{=}3 \wedge (i{=}2 \Rightarrow t'{=}t{+}1)\ \textbf{fi}$$

$$= \ i'{=}3 \wedge \textbf{if}\ i{=}1\ \textbf{then}\ t'{=}t\ \textbf{else}\ i{=}2 \Rightarrow t'{=}t{+}1\ \textbf{fi}$$

$$= \ i'{=}3 \wedge (0{<}i{\le}2 \Rightarrow t'{=}t{+}i{-}1)$$

Now I guess

$$bar_n \ = \ i'{=}3 \wedge (0{<}i{\le}n \Rightarrow t'{=}t{+}i{-}1)$$

Replacing n with ∞ produces

$$bar_\infty \ = \ i'{=}3 \wedge (0{<}i \Rightarrow t'{=}t{+}i{-}1)$$