417 Let the state consist of binary variables b and c. Let

$$W = \textbf{if } b \textbf{ then } P.\ W \textbf{ else } ok \textbf{ fi}$$

$$X = \textbf{if } b \lor c \textbf{ then } P.\ X \textbf{ else } ok \textbf{ fi}$$

(a) Find a counterexample to $W.\,X = X$.

(b) Now let W and X be the weakest solutions of those equations, and prove $W.\,X = X$.

After trying the question, scroll down to the solution.

(a) Find a counterexample to $W. X = X$.

§ Let $P = ok$, let $W = c{:=}\ b{\vee}c$, and let $X = b{:=}\ \bot$. We have to check that the given equations are satisfied. First, the W equation.

$$
\begin{array}{lll}
 & \textbf{if } b \textbf{ then } P.\ W \textbf{ else } ok \textbf{ fi} & \text{expand. } \bot \text{ is identity for } \vee \\
= & \textbf{if } b \textbf{ then } ok.\ c{:=}\ b{\vee}c \textbf{ else } c{:=}\ \bot{\vee}c \textbf{ fi} & ok \text{ is identity. In the } \textbf{else} \text{ part, } b \text{ is } \bot . \\
= & \textbf{if } b \textbf{ then } c{:=}\ b{\vee}c \textbf{ else } c{:=}\ b{\vee}c \textbf{ fi} & \text{case idempotent} \\
= & W &
\end{array}
$$

Next, check the X equation.

$$
\begin{array}{lll}
 & \textbf{if } b{\vee}c \textbf{ then } P.\ X \textbf{ else } ok \textbf{ fi} & \text{In the } \textbf{else} \text{ part, } b \text{ is } \bot . \\
= & \textbf{if } b{\vee}c \textbf{ then } ok.\ b{:=}\ \bot \textbf{ else } b{:=}\ \bot \textbf{ fi} & ok \text{ is identity, case idempotent} \\
= & X &
\end{array}
$$

So both equations are satisfied. Now we check that $W. X = X$ is not satisfied.

$$
\begin{array}{lll}
 & W. X & \\
= & c{:=}\ b{\vee}c.\ \ b{:=}\ \bot & \text{expand final assignment} \\
= & c{:=}\ b{\vee}c.\ \ \neg b' \ \wedge\ c'{=}c & \text{substitution law} \\
= & \neg b' \ \wedge\ c' = b{\vee}c &
\end{array}
$$

And

$$
\begin{array}{ll}
 & X \\
= & b{:=}\ \bot \\
= & \neg b' \ \wedge\ c'{=}c
\end{array}
$$

When the initial state is $b \wedge \neg c$, $(W. X)$ leaves c' with value $\top$, but X leaves c' with value $\bot$.

(b) Now let W and X be the weakest solutions of those equations, and prove $W. X = X$.