425 (weak data-stack) In Subsection 7.1.3 we designed a program-stack theory so weak that we could add axioms to count pushes and pops without inconsistency. Design a similarly weak data-stack theory.

After trying the question, scroll down to the solution.

§ $stack \neq null$

$push\ stack\ X\text{:}\ \ stack$

$top\ (push\ s\ x)\ =\ x$

$top\ (balance\ s)\ =\ top\ s$

$s, pop\ (balance\ (push\ s\ X))\text{: } balance\ s$

Now we can add

$count\text{: } stack \rightarrow nat$

$count\ (push\ s\ x)\ =\ count\ s + 1$

$count\ (pop\ s)\ =\ count\ s + 1$

We don't need an empty stack to start; we can just take note of the count at the start and subtract that whenever we want the relative count. Here's an implementation.

$stack = [nat;\ {*X}]$

$push\ =\ \langle s\text{: } stack\cdot \langle x\text{: } X\cdot [s\ 0 + 1];;s[1;..\#s];;[x]\rangle\rangle$

$pop\ =\ \langle s\text{: } stack\cdot [s\ 0 + 1];;s[1;..\#s-1]\rangle$

$top\ =\ \langle s\text{: } stack\cdot s(\#s-1)\rangle$

$count\ =\ \langle s\text{: } stack\cdot s\ 0\rangle$

To prove the implementation, we need to define *balance*

$balance\ =\ \langle s\text{: } stack\cdot \S t\text{: } stack\cdot s[1;..\#s]=t[1;..\#t]\rangle$

but we don't need to implement it.