485 (merged composition) Concurrent composition $P||Q$ requires that P and Q have no variables in common, although each can make use of the initial values of the other's variables by making a private copy. In this question we explore another kind of composition, let's say merged composition $P|||Q$. Like sequential composition, it requires P and Q to have the same variables. Like concurrent composition, it can be executed by executing the processes concurrently, but each process makes its assignments to local copies of variables. Then, when both processes are finished, the final value of a variable is determined as follows: if both processes left it unchanged, it is unchanged; if one process changed it and the other left it unchanged, its final value is the changed one; if both processes changed it, its final value is arbitrary. This final rewriting of variables does not require coordination or communication between the processes; each process rewrites those variables it has changed. In the case when both processes have changed a variable, we do not even require that the final value be one of the two changed values; the rewriting may mix the bits.

(a) Formally define merged composition, including time.

(b) What laws apply to merged composition?

(c) Under what circumstances is it unnecessary for a process to make private copies of state variables?

(d) In variables x , y , and z , without using $|||$, express

$$x:= z \;|||\; y:= z$$

(e) In variables x , y , and z , without using $|||$, express

$$x:= y \;|||\; y:= x$$

(f) In variables x , y , and z , without using $|||$, express

$$x:= y \;|||\; x:= z$$

(g) In variables x , y , and z , prove

$$x:= y \;|||\; x:= z \;\;=\;\; \textbf{if } x{=}y \textbf{ then } x:= z \textbf{ else if } x{=}z \textbf{ then } x:= y \textbf{ else } x:= y \;|||\; x:= z \textbf{ fi fi}$$

(h) In binary variables x , y and z , without using $|||$, express

$$x:= x \land z \;\;|||\;\; y:= y \land \neg z \;\;|||\;\; x:= x \land \neg z \;\;|||\;\; y:= y \land z$$

(i) Let w: 0,..4 and z: 0, 1 be variables. Without using $|||$, express

$$\begin{array}{ll} & w:= 2 \times (div\ w\ 2 \uparrow z) \;+\; mod\ w\ 2 \uparrow (1-z) \\ ||| & w:= 2 \times (div\ w\ 2 \uparrow (1-z)) \;+\; mod\ w\ 2 \uparrow z \end{array}$$

After trying the question, scroll down to the solution.

(a) Formally define merged composition, including time.

§
$$\begin{aligned} P|||Q \;=\; & \exists x_P, x_Q, y_P, y_Q, ..., t_P, t_Q\cdot \\ & \quad \langle x', y', ..., t' \to P\rangle\, x_P\, y_P\, ...\, t_P \\ & \wedge\ \langle x', y', ..., t' \to Q\rangle\, x_Q\, y_Q\, ...\, t_Q \\ & \wedge\ (x_P{=}x \Rightarrow x'{=}x_Q) \wedge (x_Q{=}x \Rightarrow x'{=}x_P) \\ & \wedge\ (y_P{=}y \Rightarrow y'{=}y_Q) \wedge (y_Q{=}y \Rightarrow y'{=}y_P) \\ & \wedge\ ... \\ & \wedge\ t' = t_P \uparrow t_Q \end{aligned}$$

(b) What laws apply to merged composition?

§ One law that works for merged composition but not for concurrent composition is
$$P|||ok \;=\; P \;=\; ok|||P$$

(c) Under what circumstances is it unnecessary for a process to make private copies of variables?

(d) In variables x , y , and z , without using $|||$, express
$$x{:=}\, z \;|||\; y{:=}\, z$$

§ $x'{=}y'{=}z'{=}z$

(e) In variables x , y , and z , without using $|||$, express
$$x{:=}\, y \;|||\; y{:=}\, x$$

§ $x'{=}y \wedge y'{=}x \wedge z'{=}z$

(f) In variables x , y , and z , without using $|||$, express
$$x{:=}\, y \;|||\; x{:=}\, z$$

§ $(x{=}y \Rightarrow x'{=}z) \wedge (x{=}z \Rightarrow x'{=}y) \wedge y'{=}y \wedge z'{=}z$

(g) In variables x , y , and z , prove
$$x{:=}\, y \;|||\; x{:=}\, z \;=\; \textbf{if } x{=}y \textbf{ then } x{:=}\, z \textbf{ else if } x{=}z \textbf{ then } x{:=}\, y \textbf{ else } x{:=}\, y \;|||\; x{:=}\, z \textbf{ fi fi}$$

§
$$\begin{aligned} & \textbf{if } x{=}y \textbf{ then } x{:=}\, z \textbf{ else if } x{=}z \textbf{ then } x{:=}\, y \textbf{ else } x{:=}\, y \;|||\; x{:=}\, z \textbf{ fi fi} \\ & \hspace{20em} \text{using the answer for (f)} \\ = \; & \quad (x{=}y \Rightarrow x'{=}z \wedge y'{=}y \wedge z'{=}z) \wedge (x{\ne}y \wedge x{=}z \Rightarrow x'{=}y \wedge y'{=}y \wedge z'{=}z) \\ & \wedge\ (x{\ne}y \wedge x{\ne}z \Rightarrow (x{=}y \Rightarrow x'{=}z) \wedge (x{=}z \Rightarrow x'{=}y) \wedge y'{=}y \wedge z'{=}z) \\ & \hspace{20em} \text{a few steps of binary algebra} \\ = \; & (x{=}y \Rightarrow x'{=}z) \wedge (x{=}z \Rightarrow x'{=}y) \wedge y'{=}y \wedge z'{=}z \qquad \text{using the answer for (f) again} \\ = \; & x{:=}\, y \;|||\; x{:=}\, z \end{aligned}$$

(h) In binary variables x , y and z , without using $|||$, express
$$x{:=}\, x{\wedge}z \;|||\; y{:=}\, y{\wedge}\neg z \;|||\; x{:=}\, x{\wedge}\neg z \;|||\; y{:=}\, y{\wedge}z$$

§
$$\begin{aligned} & x{:=}\, x{\wedge}z \;|||\; y{:=}\, y{\wedge}\neg z \;|||\; x{:=}\, x{\wedge}\neg z \;|||\; y{:=}\, y{\wedge}z \\ = \; & \textbf{if } z \textbf{ then } x{:=}\, x \;|||\; y{:=}\, \bot \;|||\; x{:=}\, \bot \;|||\; y{:=}\, y \\ & \textbf{else } x{:=}\, \bot \;|||\; y{:=}\, y \;|||\; x{:=}\, x \;|||\; y{:=}\, \bot \textbf{ fi} \\ = \; & x{:=}\, \bot \;|||\; y{:=}\, \bot \\ = \; & \neg x' \wedge \neg y' \wedge z'{=}z \end{aligned}$$

(i) Let $w{:}\ 0,..4$ and $z{:}\ 0, 1$ be variables. Without using $|||$, express
$$\begin{aligned} & \quad w{:=}\, 2 \times (div\ w\ 2){\uparrow}z \;+ (mod\ w\ 2){\uparrow}(1{-}z) \\ & |||\quad w{:=}\, 2 \times (div\ w\ 2){\uparrow}(1{-}z) \;+ (mod\ w\ 2){\uparrow}z \end{aligned}$$

§
$$\begin{aligned} & \quad w{:=}\, 2 \times (div\ w\ 2){\uparrow}z \;+\; (mod\ w\ 2){\uparrow}(1{-}z) \\ & |||\quad w{:=}\, 2 \times (div\ w\ 2){\uparrow}(1{-}z) \;+\; (mod\ w\ 2){\uparrow}z \\ = \; & \textbf{if } z{=}0 \textbf{ then } w{:=}\, 2 \times div\ w\ 2 \;+\; 1 \;|||\; w{:=}\, 2 + mod\ w\ 2 \end{aligned}$$

$\quad$ **else** $w{:=}\ 2 + mod\ w\ 2 \ \ ||| \ \ w{:=}\ 2 \times div\ w\ 2 \ + \ 1$ **fi** $\qquad ||| \ \text{is symmetric}$

$=\quad w{:=}\ 2 \times div\ w\ 2 \ + \ 1 \ \ ||| \ \ w{:=}\ 2 + mod\ w\ 2$

$=\quad$ **if** $w{=}0$ **then** $w{:=}\ 1 \ ||| \ w{:=}\ 2$

$\quad$ **else if** $w{=}1$ **then** $w{:=}\ 1 \ ||| \ w{:=}\ 3$

$\quad$ **else if** $w{=}2$ **then** $w{:=}\ 3 \ ||| \ w{:=}\ 2$

$\quad$ **else** $w{:=}\ 3 \ ||| \ w{:=}\ 3$ **fi fi fi**

$=\quad (w{\neq}0 \Rightarrow w'{=}3) \land z'{=}z$