X5.0 Angel and Demon play a game of alternating moves. To start, Demon creates n piles of sand: pile 0, pile 1, ..., pile (n–1). Angel is trying to make all n piles empty. Demon is trying to prevent that for as long as possible, preferably forever. For her turn, Angel removes one grain of sand from one of the piles. For his turn, Demon adds as much sand as he wants to all the other nonempty piles. Demon cannot add sand to the pile that Angel has just removed a grain from, nor to an empty pile. Informally, the game is

while (there is a nonempty pile)
do (Angel removes one grain from some nonempty pile,
and Demon adds as much sand as he wants to all other nonempty piles).
t:= t+1 **od**

Does execution terminate?

After trying the question, scroll down to the solution.

§ Formally, let S: n*nat be a state variable telling the number of grains in each pile. Here is the game (quantifications are over $0,..n$).

while $(\exists i\cdot\ S_i > 0)$
do $(\exists i\cdot\ S'_i = S_i - 1 \geq 0\ \wedge\ (\forall j\cdot\ j \neq i\ \Rightarrow\ S'_j = S_j = 0\ \vee\ S'_j \geq S_j > 0)) \wedge t' = t+1$ **od**

Angel has a strategy. Keep taking a grain from pile 0 until it is empty. Then keep taking a grain from pile 1 until it is empty. Then from pile 2, and so on, in order. We now refine the body of the loop as follows.

$(\exists i\cdot\ S'_i = S_i - 1 \geq 0\ \wedge\ (\forall j\cdot\ j \neq i\ \Rightarrow\ S'_j = S_j = 0\ \vee\ S'_j \geq S_j > 0)) \wedge t' = t+1$
$\Leftarrow$ $(\exists i\cdot\ (\forall j\colon 0,..i\cdot\ S'_j = S_j = 0)\ \wedge\ S'_i = S_i - 1 \geq 0\ \wedge\ (\forall j\colon i+1,..n\cdot\ S'_j \geq S_j > 0)) \wedge t' = t+1$

Does this work? Consider some particular pile of sand ("this" pile). While Angel is working on a previous pile, Demon adds sand to this pile. When the immediately previous pile becomes empty, Demon adds sand to this pile for the last time. After that, Demon cannot affect this pile, first because Angel is working on it, then because it is empty. The amount of sand in this pile just after Demon's last time adding to it is the sum of the finite amounts that Demon has added to it. And that determines how long Angel will spend emptying this pile.

Let T: n*nat be a constant defined as follows. T_0 is the number of grains chosen by Demon for pile 0 initially. T_{i+1} is the number of grains of sand in pile $i+1$ when pile i becomes empty. Introducing T gives us a way to refer to Demon's choices; it is not a constraint on Demon's choices. The execution time is

$\Sigma i\cdot\ T_i$

T_0 is just S_0 initially and T_{i+1} is just S_{i+1} at some other particular instant. Demon could just as well make his final choices as the initial value of S , and then retire from the game, leaving Angel to work through the piles with execution time $\Sigma i\cdot\ S_i$.