X7.0 Let a , c , and x be natural variables. Variables a and c are implementer's variables, and x is a user's variable for the operations

$$start \;=\; a:= 1.\; c:= 0$$
$$double \;=\; a:= a\times 2.\; c:= c+1$$
$$ask \;=\; x:= c$$

Operation *start* starts variable a at 1 . Then repeated use of operation *double* doubles it some number of times. Variable c counts how many times a is doubled. Operation *ask* asks how many times a has been doubled since the last *start* operation. Reimplement this theory replacing the old implementer's variable a with nothing.

(a) What is the data transformer? Prove it is a data transformer.

(b) Using your data transformer, transform *double* .

After trying the question, scroll down to the solution.

(a) What is the data transformer? Prove it is a data transformer.

§ One data transformer is

$$\top$$

We need to prove $\forall w\cdot \exists v\cdot D$ where w are the new variables, v are the old variables, and D is the data transformer. There are no new implementer's variables, so there is no $\forall$ quantifier. We are replacing only a, so we prove

$$\begin{array}{lll} & \exists a\cdot \top & \text{idempotent} \\ = & \top & \end{array}$$

Another data transformer is

$$2^c = a$$

We prove

$$\begin{array}{lll} & \exists a\cdot 2^c = a & \text{identity} \\ = & \exists a\cdot 2^c = a \;\wedge\; \top & \text{one-point on } a \\ = & \top & \end{array}$$

(b) Using your data transformer, transform *double* .

§ Using data transformer $\top$,

$$\begin{array}{lll} & \forall a\cdot \top \Rightarrow \exists a'\cdot \top \wedge double & \\ = & \forall a\cdot \exists a'\cdot double & \\ = & \forall a\cdot \exists a'\cdot (a:= a\times 2.\; c:= c+1) & \\ = & \forall a\cdot \exists a'\cdot a' = a\times 2 \;\wedge\; c' = c+1 \;\wedge\; x'=x & \text{one-point on } a' \\ = & \forall a\cdot c' = c+1 \;\wedge\; x'=x & \text{unused } a \\ = & c' = c+1 \;\wedge\; x'=x & \text{assignment} \\ = & c:= c+1 & \end{array}$$

Using data transformer $2^c = a$,

$$\begin{array}{lll} & \forall a\cdot 2^c = a \;\Rightarrow\; \exists a'\cdot 2^{c'} = a' \wedge double & \\ = & \forall a\cdot 2^c = a \;\Rightarrow\; \exists a'\cdot 2^{c'} = a' \wedge (a:= a\times 2.\; c:= c+1) & \\ = & \forall a\cdot 2^c = a \;\Rightarrow\; \exists a'\cdot 2^{c'} = a' \wedge\; a' = a\times 2 \;\wedge\; c' = c+1 \;\wedge\; x'=x & \text{one-point on } a' \\ = & \forall a\cdot 2^c = a \;\Rightarrow\; 2^{c'} = a\times 2 \;\wedge\; c' = c+1 \;\wedge\; x'=x & \text{one-point on } a \\ = & 2^{c'} = 2^c\times 2 \;\wedge\; c' = c+1 \;\wedge\; x'=x & \text{context} \\ = & 2^{c+1} = 2^c\times 2 \;\wedge\; c' = c+1 \;\wedge\; x'=x & \text{arithmetic} \\ = & c' = c+1 \;\wedge\; x'=x & \text{assignment} \\ = & c:= c+1 & \end{array}$$