X7.2 In Data-Queue Theory, prove that if we start with an empty queue, then join x, then join any number of y's, the front will still be x.

After trying the question, scroll down to the solution.

§ We need a notation for joining any number of y 's. Mathematicians use exponents to indicate a repetition of function application, like this:

$$join^0\ q\ y\ =\ q$$
$$join^{n+1}\ q\ y\ =\ join\ (join^n\ q\ y)\ y$$

But that is a misuse of notation, since it doesn't obey the laws of exponents. For example,

$$join^0 = 1$$

I suppose a mathematician would groan and say “be reasonable”. But it's easy to not cause any problem by just choosing another notation. How about a pre-superscript:

$${}^{0}join\ q\ y\ =\ q$$
$${}^{n+1}join\ q\ y\ =\ join\ ({}^{n}join\ q\ y)\ y$$

Now we want to prove

$$\forall n\colon nat\cdot\ front\ ({}^{n}join\ (join\ empty\ x)\ y)\ =\ x$$

We need nat induction. I choose the form I am most familiar with:

$$P\ 0 \wedge \forall n\colon nat\cdot P\ n \Rightarrow P\ (n{+}1)\ \ \Longrightarrow\ \ \forall n\colon nat\cdot P\ n$$

where

$$P\ =\ \langle n\colon nat\cdot\ front\ ({}^{n}join\ (join\ empty\ x)\ y)\ =\ x\rangle$$

$P\ 0$

$=\quad front\ ({}^{0}join\ (join\ empty\ x)\ y)\ =\ x$ — definition of ${}^{0}join$

$=\quad front\ (join\ empty\ x)\ =\ x$ — queue axiom

$=\quad \top$

$P\ (n{+}1)$

$=\quad front\ ({}^{n+1}join\ (join\ empty\ x)\ y)\ =\ x$ — definition of ${}^{n+1}join$

$=\quad front\ (join\ ({}^{n}join\ (join\ empty\ x)\ y)\ y)\ =\ x$

I am aiming for $P\ n$, which is

$$front\ ({}^{n}join\ (join\ empty\ x)\ y)\ =\ x$$

so I need to get rid of one $join$, and that's what the data-queue axiom

$$q{\neq}empty\ \Rightarrow\ front\ (join\ q\ x) = front\ q$$

does. It requires the antecedent $q{\neq}empty$, which in this context is

$${}^{n}join\ (join\ empty\ x)\ y \neq empty$$

So I'll take a break from the proof to prove this lemma by nat induction. Again, I use the form

$$P\ 0 \wedge \forall n\colon nat\cdot P\ n \Rightarrow P\ (n{+}1)\ \ \Longrightarrow\ \ \forall n\colon nat\cdot P\ n$$

where

$$P\ =\ \langle n\colon nat\cdot\ {}^{n}join\ (join\ empty\ x)\ y \neq empty\rangle$$

$P\ 0$

$=\quad {}^{0}join\ (join\ empty\ x)\ y \neq empty$ — definition of ${}^{0}join$

$=\quad join\ empty\ x \neq empty$ — data-queue axiom $join\ q\ x \neq empty$

$=\quad \top$

$P\ (n{+}1)$

$=\quad {}^{n+1}join\ (join\ empty\ x)\ y \neq empty$ — definition of ${}^{n+1}join$

$=\quad join\ ({}^{n}join\ (join\ empty\ x)\ y)\ y \neq empty$ — data-queue axiom $join\ q\ x \neq empty$

$=\quad \top$

So now we have proven the lemma $\forall n\colon nat\cdot\ {}^{n}join\ (join\ empty\ x)\ y \neq empty$, and we can return to the proof that we took a break from.

$P\ (n{+}1)$

$=\quad front\ ({}^{n+1}join\ (join\ empty\ x)\ y)\ =\ x$ definition of ${}^{n+1}join$

$=\quad front\ (join\ ({}^{n}join\ (join\ empty\ x)\ y)\ y)\ =\ x$ use the lemma and the axiom

$=\quad front\ ({}^{n}join\ (join\ empty\ x)\ y)\ =\ x$

$=\quad P\ n$

We have proven $P\ 0$ and $\forall n: nat\cdot\ P\ n \Rightarrow P\ (n+1)$ so by induction we have proven

$$\forall n: nat\cdot\ P\ n$$

which is

$$\forall n: nat\cdot\ front\ ({}^{n}join\ (join\ empty\ x)\ y)\ =\ x$$

as required.