CSC373— Algorithm Design, Analysis, and Complexity — Spring 2016

Proof of the Flow Value Lemma Revisited.
Given a max-flow problem $G = (V, E, c)$.
Defn: For any flow f, the value of the flow is $v(f) = \sum_{(s,v)\in E} f((s, v))$.
Defn: For any flow f and $s - t$ cut (A, B), the net flow (from A to B) is

$$N_f(A) = \sum_{\{(u,v)\in E \mid u\in A, v\in B\}} f((u, v)) - \sum_{\{(u,v)\in E \mid u\in B, v\in A\}} f((u, v)).$$

Flow Value Lemma: $v(f) = N_f(A)$.
Proof:

$$\begin{aligned}
v(f) &= \sum_{(s,u)\in E} f((s, u)), \text{ by definition,} \\
&= \sum_{(s,u)\in E} f((s, u)) + \sum_{v\in A\setminus\{s\}} \left[\sum_{(v,u)\in E} f((v, u)) - \sum_{(u,v)\in E} f((u, v)) \right], \text{ by flow conservation at } v \in A\setminus\{s\}, \\
&= \sum_{v\in A} \left[\sum_{(v,u)\in E} f((v, u)) - \sum_{(u,v)\in E} f((u, v)) \right]. \qquad (1)
\end{aligned}$$

In the last row above we have included $v = s$ in the sum over v and absorbed the first sum of the outflow from s. We can do this since, when $v = s$, there are only out-bound edges of the form $(s, u) \in E$ and no in-bound edges (u, s). (In the lecture I missed this last step and, by missing it, I complicated things later.)

The first term on the right hand side of (1) above can be expanded as:

$$\sum_{v\in A} \sum_{(v,u)\in E} f((v, u)) = \sum_{v\in A} \left[\sum_{\{(v,u)\in E \mid u\in A\}} f((v, u)) \right] + \sum_{v\in A} \left[\sum_{\{(v,u)\in E \mid u\in B\}} f((v, u)) \right]. \qquad (2)$$

Similarly, the second term on the right hand side of (1) above can be expanded as:

$$\sum_{v\in A} \sum_{(u,v)\in E} f((u, v)) = \sum_{v\in A} \left[\sum_{\{(u,v)\in E \mid u\in A\}} f((u, v)) \right] + \sum_{v\in A} \left[\sum_{\{(u,v)\in E \mid u\in B\}} f((u, v)) \right]. \qquad (3)$$

Note that the first terms on the right hand side of (2) and (3) are the same, that is,

$$\sum_{v\in A} \left[\sum_{\{(v,u)\in E \mid u\in A\}} f((v, u)) \right] = \sum_{v\in A} \left[\sum_{\{(u,v)\in E \mid u\in A\}} f((u, v))) \right] = \sum_{\{(x,y)\in E \mid x\in A, y\in A\}} f((x, y)). \qquad (4)$$

As a consequence these two terms cancel from (1) and, by equations (2), (3) and (4), we have

$$\begin{aligned}
v(f) &= \sum_{v\in A} \left[\sum_{(v,u)\in E} f((v, u)) - \sum_{(u,v)\in E} f((u, v)) \right], \text{ by (1),} \\
&= \sum_{v\in A} \left[\sum_{\{(v,u)\in E \mid u\in B\}} f((v, u)) \right] - \sum_{v\in A} \left[\sum_{\{(u,v)\in E \mid u\in B\}} f((u, v)) \right], \text{ by (2-4),} \\
&= N_f(A), \text{ by the definition of net flow.} \qquad (5)
\end{aligned}$$