Consider the integral

$$I_n = \int_0^1 x^n \mathrm{e}^{x-1}\, dx \tag{1}$$

where n is a non-negative integer. Note that $x^n \mathrm{e}^{x-1} > 0$ for all non-negative integers n and for all $x \in (0,1)$. Therefore, $I_n > 0$ for all non-negative integers n. Moreover, $x^n > x^{n+1}$ for all non-negative integers n and for all $x \in (0,1)$. Hence, $x^n \mathrm{e}^{x-1} > x^{n+1}\mathrm{e}^{x-1}$ for all non-negative integers n and for all $x \in (0,1)$. Therefore, $I_n > I_{n+1}$ for all non-negative integers n. That is, the sequence $I_0, I_1, I_2, \ldots$ is positive and monotonically decreasing.

One way to compute I_n for any non-negative integer n is as follows. Note that

$$I_0 = \int_0^1 \mathrm{e}^{x-1}\, dx = \left[\mathrm{e}^{x-1}\right]_0^1 = 1 - \mathrm{e}^{-1} = 1 - 1/\mathrm{e} \tag{2}$$

Moreover, for $n \geq 1$, integrating (1) by parts leads to

$$I_n = \int_0^1 x^n \mathrm{e}^{x-1}\, dx = \left[x^n \mathrm{e}^{x-1}\right]_0^1 - \int_0^1 n x^{n-1} \mathrm{e}^{x-1}\, dx = 1 - nI_{n-1} \tag{3}$$

Therefore, we can use the recurrence

$$\begin{aligned} I_0 &= 1 - 1/\mathrm{e} \\ I_n &= 1 - nI_{n-1} \qquad \text{for } n = 1, 2, 3, \ldots \end{aligned} \tag{4}$$

to evaluate I_n for any non-negative integer n. You can use the MatLab function exp(1.0) to compute e accurately in (4). If you use (4) in a little MatLab program to compute I_n for $n = 0, 1, \ldots, 25$, you get the values listed on the next page.

You should try to write a little MatLab program yourself to compute I_n for $n = 0, 1, \ldots, 25$.

n	I_n
0	6.3212e-01
1	3.6788e-01
2	2.6424e-01
3	2.0728e-01
4	1.7089e-01
5	1.4553e-01
6	1.2680e-01
7	1.1238e-01
8	1.0093e-01
9	9.1612e-02
1	8.3877e-02
11	7.7352e-02
12	7.1773e-02
13	6.6948e-02
14	6.2731e-02
15	5.9034e-02
16	5.5459e-02
17	5.7192e-02
18	−2.9454e-02
19	1.5596e+00
20	−3.0192e+01
21	6.3504e+02
22	−1.3970e+04
23	3.2131e+05
24	−7.7114e+06
25	1.9279e+08

The values for I_n look reasonable for $n = 0, 1, 2, \ldots, 15$ or so, but the values are clearly wrong for $n = 20, 21, \ldots, 25$, since the computed I_n values are not all positive and monotonically decreasing, as they should be.

1. Explain why my MatLab program computes such inaccurate values for I_n for $n = 20, 21, \ldots, 25$.

   My program is a correct implementation of the recurrence (4). That is, the inaccuracy in the table above is not a result of a programming bug. The problem is with the recurrence (4) itself and the rounding errors that occur when you implement it in floating-point arithmetic.

   In this explanation, it is not sufficient to say that there is round-off error in the computation. Although this is true and should play a part in your explanation, there is round-off error in almost all floating-point computations and most of them produce accurate results. You need to explain why the round-off error produces such bad results in this case.

2. Re-arrange the recurrence

$$I_n = 1 - nI_{n-1}$$

starting it from a different initial value so that your new recurrence computes accurate values for I_n for $n = 0, 1, 2, \ldots, 25$.

Explain why you believe your new method produces accurate results.

Write a little MatLab program that uses your new method to compute I_n for $n = 0, 1, 2, \ldots, 25$. Your program should also print n and I_n for $n = 0, 1, 2, \ldots, 25$ in a nicely formatted table.