This assignment is due at the **start** of the tutorial on Friday, 18 November 2011.

1. [4 marks: 2 marks for each part]

   I mentioned in class that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ for all vectors $x \in \mathbb{R}^n$ and all integers $n \ge 1$.

   A similar result does not hold for matrices. To illustrate this,

   (a) find a 2×2 matrix A such $\|A\|_\infty < \|A\|_2$;

   (b) find another 2×2 matrix B such that $\|B\|_\infty > \|B\|_2$.

2. [5 marks]

   We showed in class that, if $Ax = b$ and $A\hat{x} = \hat{b}$, where x, b, $\hat{x}$ and $\hat{b} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, then

$$\frac{\|x - \hat{x}\|}{\|x\|} \le \text{cond}(A)\, \frac{\|b - \hat{b}\|}{\|b\|} \tag{1}$$

   I mentioned in class that you can also show that

$$\frac{1}{\text{cond}(A)}\, \frac{\|b - \hat{b}\|}{\|b\|} \le \frac{\|x - \hat{x}\|}{\|x\|} \tag{2}$$

   Therefore, combining (1) and (2), we get

$$\frac{1}{\text{cond}(A)}\, \frac{\|b - \hat{b}\|}{\|b\|} \le \frac{\|x - \hat{x}\|}{\|x\|} \le \text{cond}(A)\, \frac{\|b - \hat{b}\|}{\|b\|} \tag{3}$$

   Show that (2) holds.

   Note that all the vector norms in (1), (2) and (3) are the same and that the matrix norm associated with cond is the matrix norm *subordinate* to that vector norm. (See page 55 of your textbook for the definition of a matrix norm *subordinate* to a vector norm if you don't recall the definition.)

3. [12 marks: 2 marks for each part]

   Suppose that both sides of a system of linear equations $Ax = b$, where A is a nonsingular matrix, are multiplied by a nonzero real number α. That is, we form the new equation $\hat{A}\hat{x} = \hat{b}$, where $\hat{A} = \alpha A$ and $\hat{b} = \alpha b$.

   (a) Does this change the true solution x of $Ax = b$? (I.e., is $x = \hat{x}$?)
   Justify your answer.

   (b) Does this change the residual $r = b - A\tilde{x}$, where $\tilde{x}$ is an approximate solution of $Ax = b$? (I.e., is $r = \hat{r}$, where $\hat{r} = \hat{b} - \hat{A}\tilde{x}$?)
   Justify your answer.

(c) Does this change the relative residual $\|r\|/\|b\|$, where r is the residual from part (b) and b is right side of $Ax = b$? (I.e., is $\|r\|/\|b\| = \|\hat{r}\|/\|\hat{b}\|$, where $\hat{r}$ is defined in (b) above?) Justify your answer.

(d) Does this change the conditioning of the system $Ax = b$? (I.e., is $\text{cond}(A) = \text{cond}(\hat{A})$?) Justify your answer.

(e) What conclusions can be drawn from parts (a)–(d) above about assessing the quality of a computed solution $\hat{x}$ of $Ax = b$?

(f) Assuming that A has an LU factorization, how does multiplying A by a nonzero real number α change the LU factorization of A?
That is, if $\hat{A} = \alpha A$, $LU = A$ and $\hat{L}\hat{U} = \hat{A}$, where L and $\hat{L}$ are unit lower triangular matrices and U and $\hat{U}$ are upper triangular matrices, how are L and $\hat{L}$ related and how are U and $\hat{U}$ related?
Justify your answer.

4. [4 marks: 2 marks for each part]

Suppose that both sides of a system of linear equations $Ax = b$, where A is a nonsingular matrix, are pre-multiplied by a nonsingular diagonal matrix D of the same dimensions as A.

(a) Does this change the true solution x of $Ax = b$? (I.e., how does the solution $\hat{x}$ of $\hat{A}\hat{x} = \hat{b}$ relate to the solution x of $Ax = b$, where $\hat{A} = DA$ and $\hat{b} = Db$?)
Justify your answer.

(b) Can this affect the conditioning of the system? (I.e., is $\text{cond}(A) = \text{cond}(\hat{A})$, where $\hat{A} = DA$?)
Justify your answer.

5. [10 marks: 5 marks for each part]

(a) Compute the LU factorization (without pivoting) of the matrix

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 4 \\ -1 & -1 & 2 \end{pmatrix}.$$

That is, compute a unit lower triangular matrix L (i.e., $L_{ii} = 1$ for all i and $L_{ij} = 0$ for all $i < j$) and an upper triangular matrix U (i.e., $U_{ij} = 0$ for all $i > j$) such that $A = LU$.

(b) Use the LU factorization from part (a) to solve $Ax = b$ for

$$b = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix}.$$