Machine Learning I
MATH80629A

Apprentissage Automatique I
MATH80629

Sequential Decision Making I
— Week #12

Today

- Motivation and introduction
  - Toward Reinforcement learning
- Planning
  - Markov Decision Process (MDP)
    - Value iteration
    - Policy iteration
- Next week: Reinforcement learning

Reinforcement Learning
Motivation

Recall

Three main components

- Task (T)
- Performance measure (P)
- Experience (E)

Supervised learning

- Experience a fixed data set
- Fit a model using this data
- Use the model to make predictions about unseen data (and to understand the data)
- Predictions may be used downstream to inform decision-making (e.g., Operations Research)

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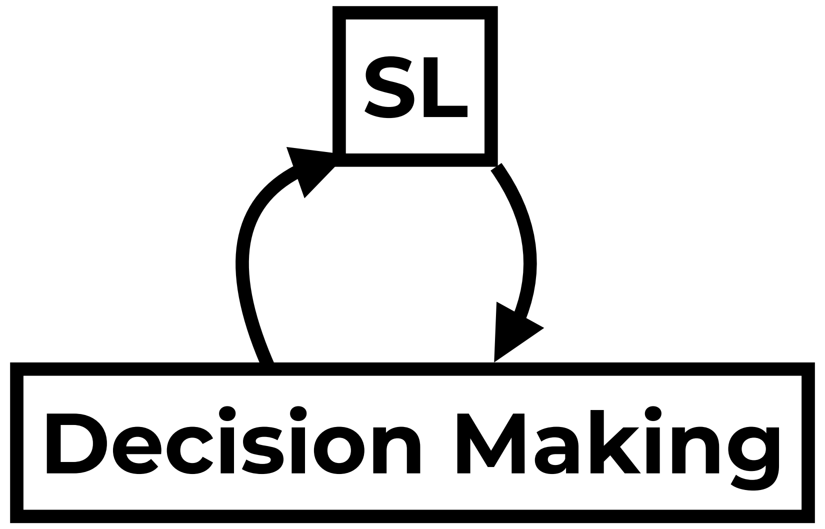
An example of learning and decision making

- Imagine building a robot that must navigate autonomously
  - The robot has wheels and a camera
- You think about using a two-stage approach:
  1. Use supervised learning to identify objects in scenes
  2. Given scene content have a decision-making module that controls its wheels

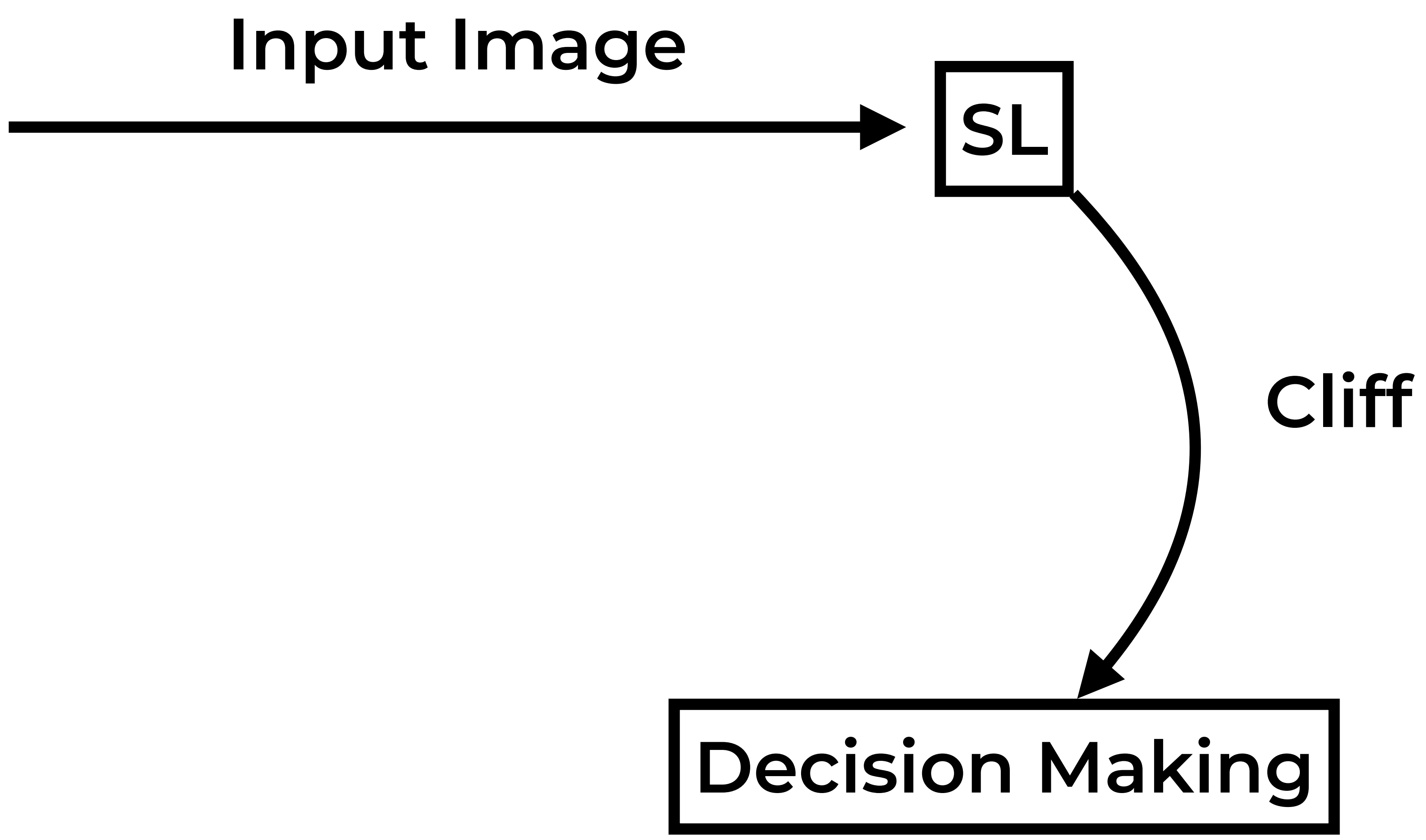


View from the robot's camera

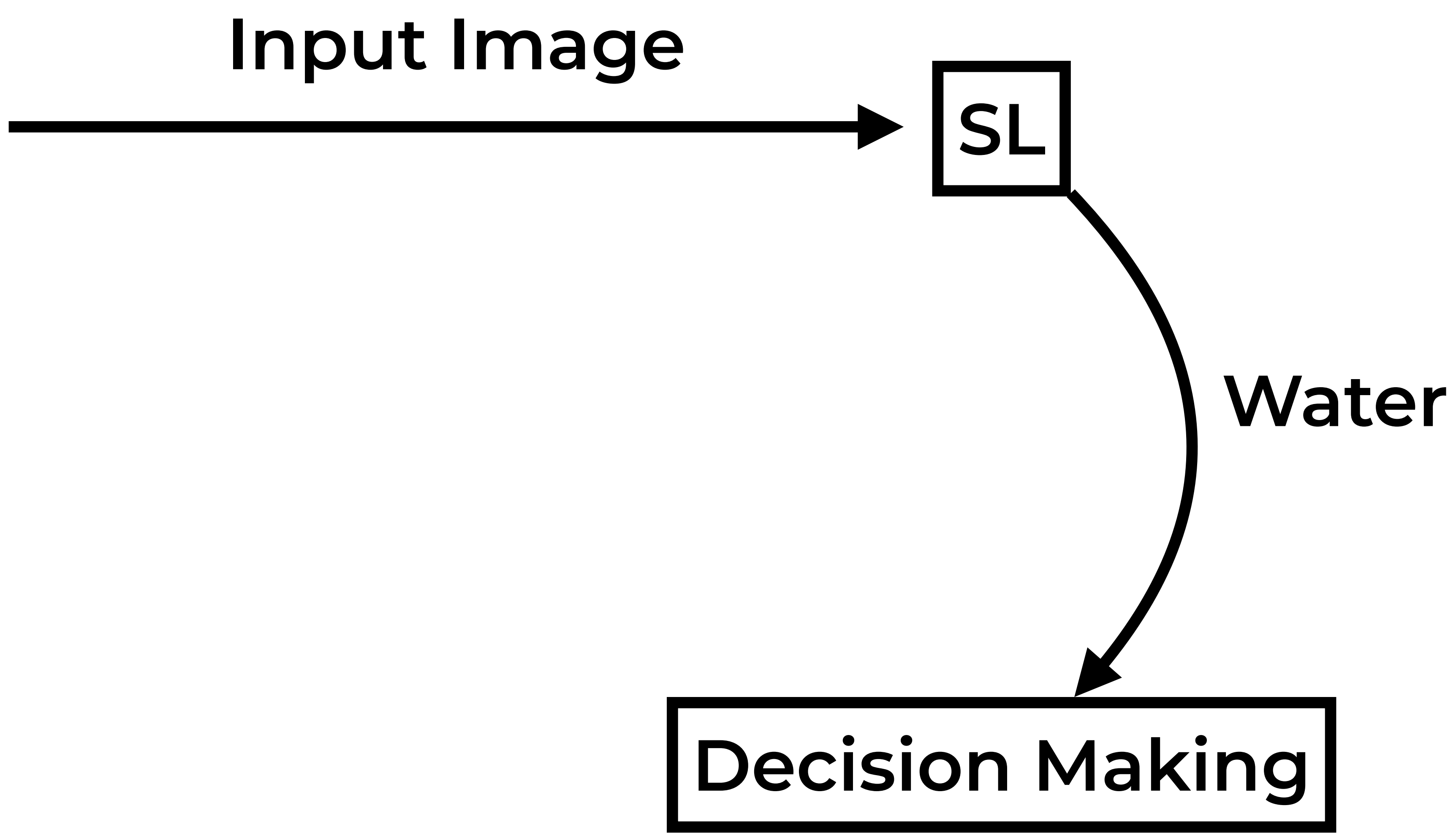
View from the robot's camera

Input Image

View from the robot’s camera



View from the robot’s camera



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  - Its objective is, for example, to maximize accuracy

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  - E.g., missing the cliff could have dire consequences. missing sky less so.
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- Supervised learning doesn't know about the decision-making
  - Its objective is, for example, to maximize accuracy
- For decision making, different errors have different costs
  - E.g., missing the cliff could have dire consequences. missing sky less so.
  - Incorporating these costs into the learning objective is tough
- Several other limitations:
  - need labeled data
  - improvements in SL do not necessarily lead to improvements in decision making
  - ...

Alternative: Reinforcement learning (RL)

- Incorporates both stages in a single framework
- Incorporates the ideas of:
  - state (observation)
  - action
  - reward

Planning:
A first step towards reinforcement learning

Alternative: Reinforcement learning (RL)

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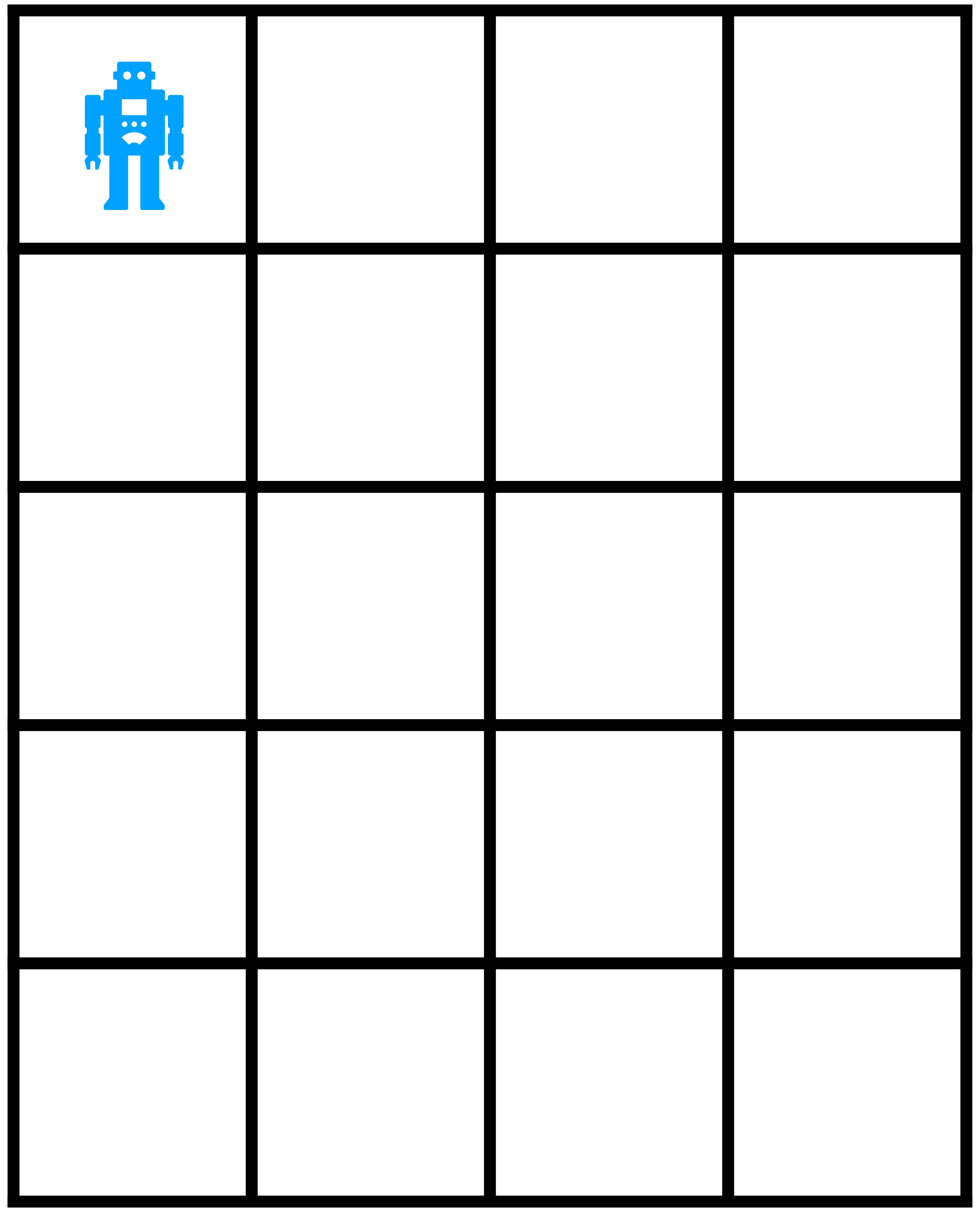
Initial example with grid world



- Each cell is a state (S)
- Actions indicate which movements are possible: $A := \{L, R, U, D\}$
- Rewards encode the task: $R(s)$
- Transition probabilities encode the outcome of an action: $P(s' \mid s, a)$

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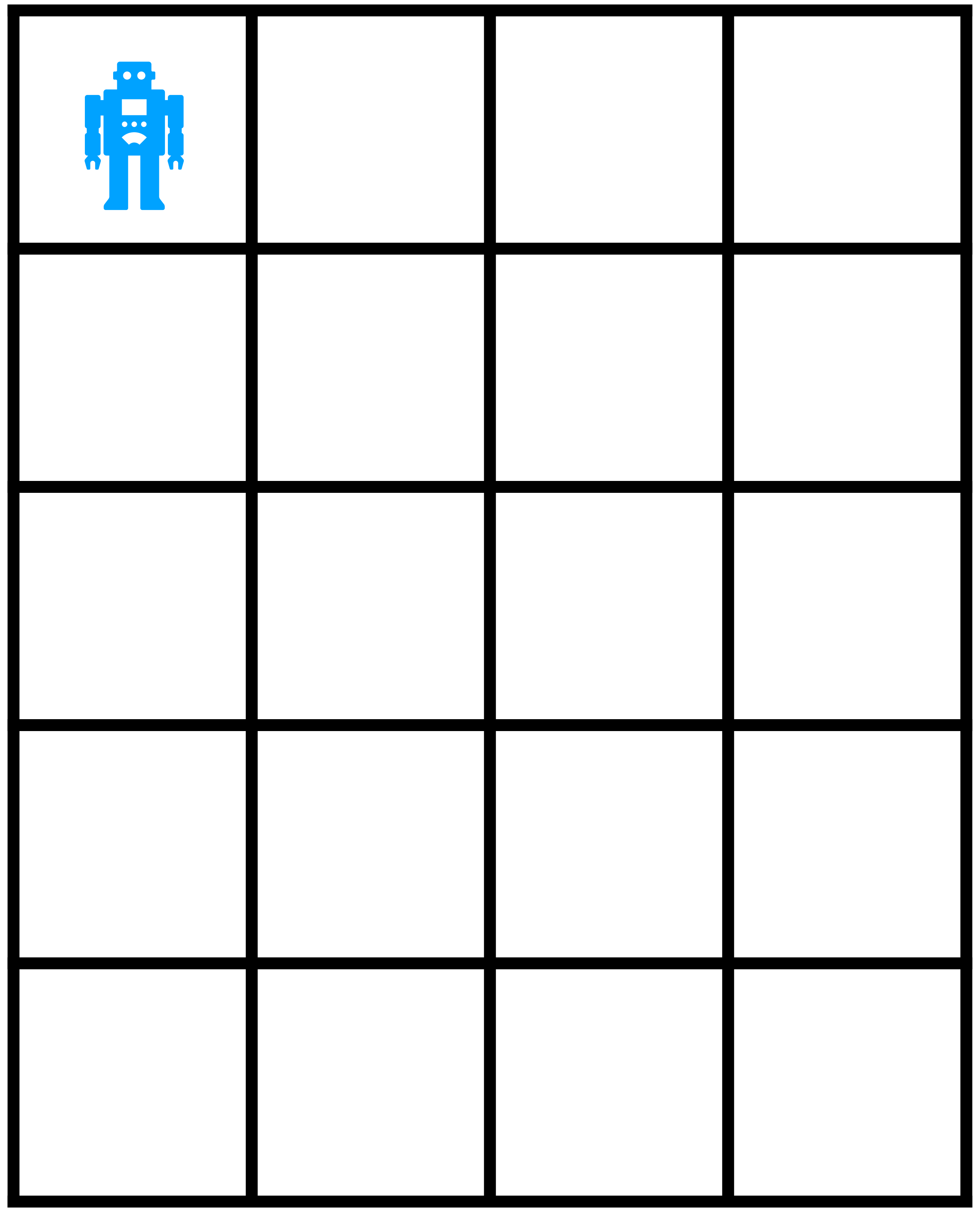

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Planning

This week we discuss a version of RL where these are observed

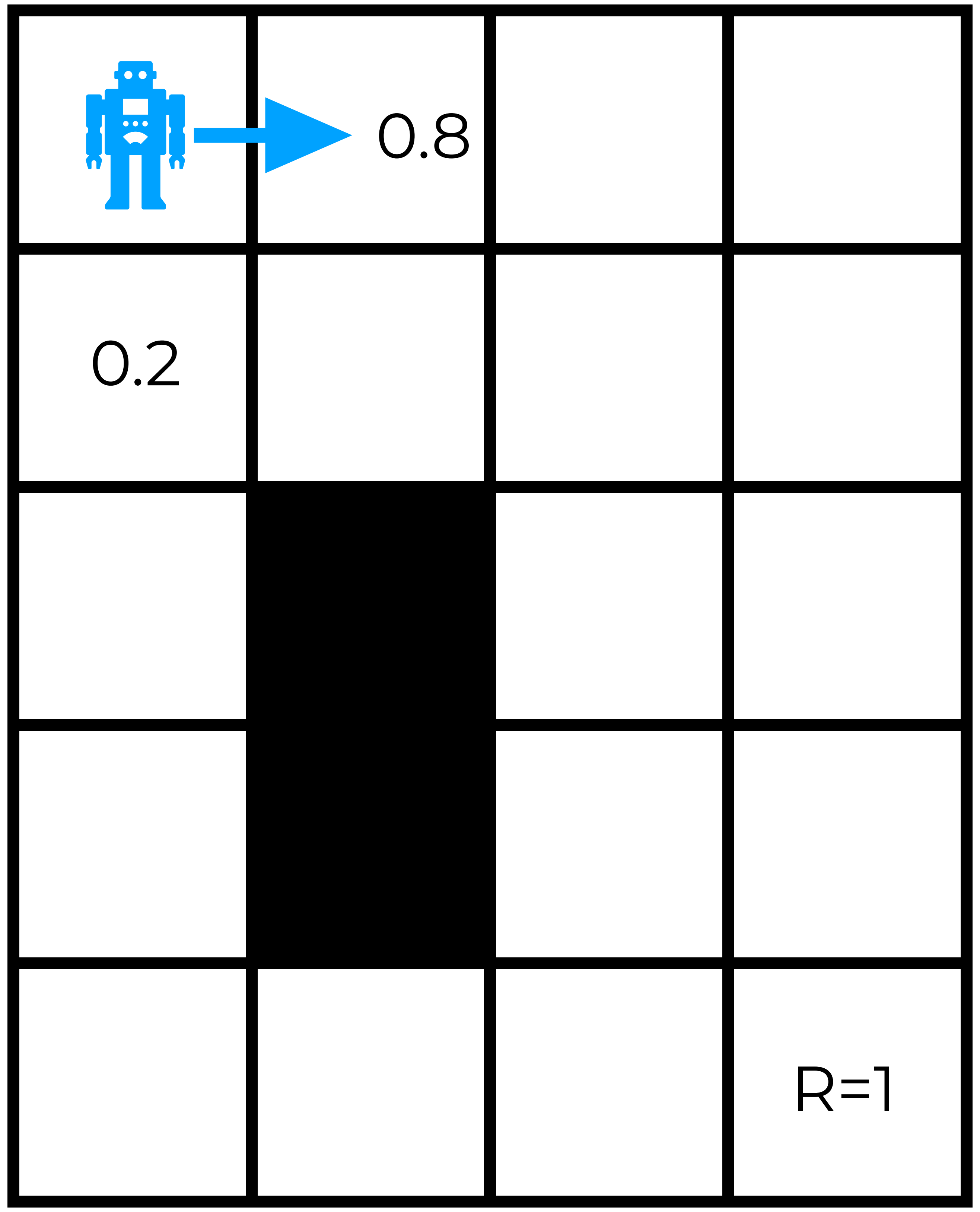
- 20 states. Start state is top-left
  - Bottom right is absorbing

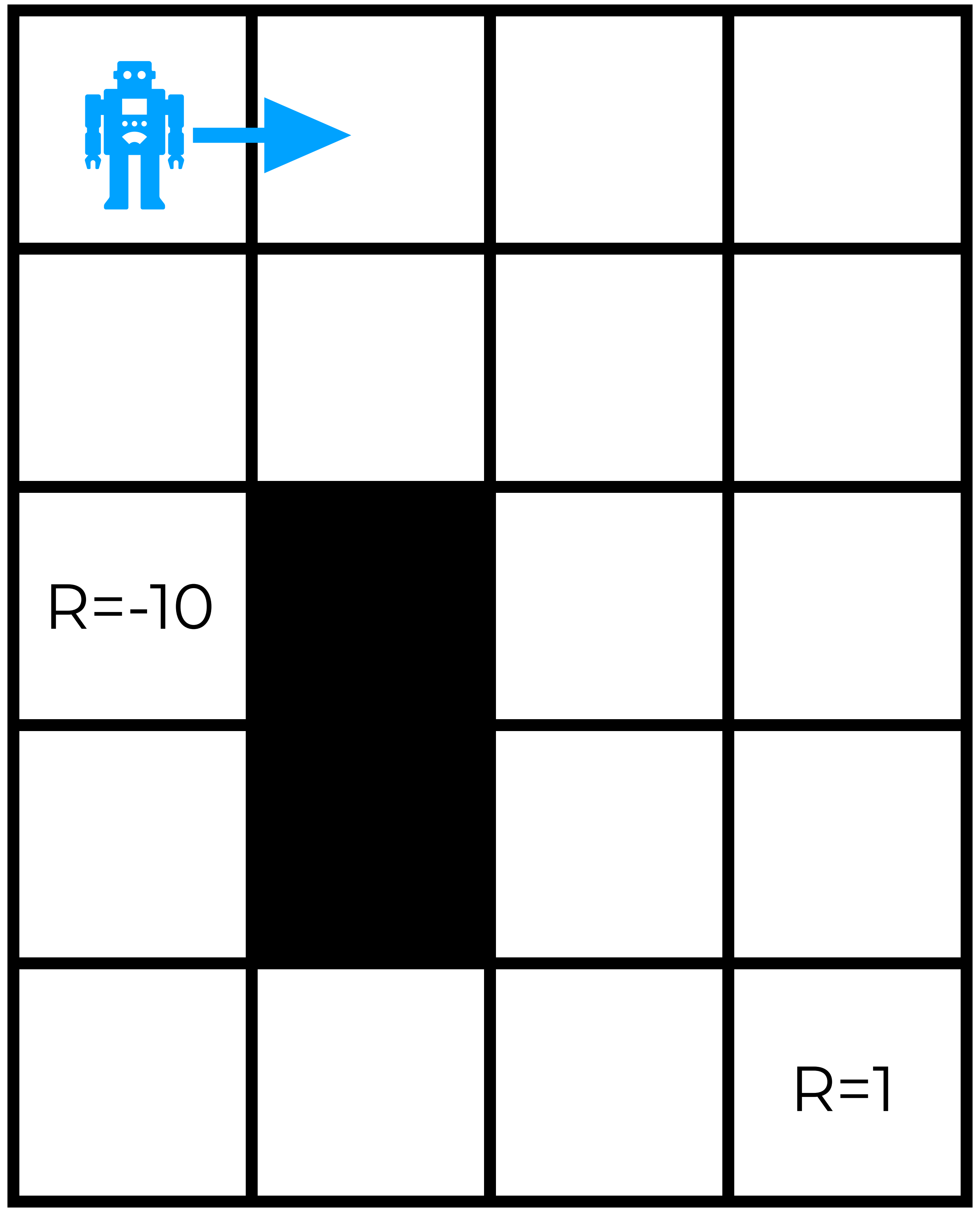
$$P(s'|s_{\text{absorbing}}, a) = \begin{cases} 1 & \text{if } s' = s, \\ 0 & \text{otherwise.} \end{cases}$$

- All rewards are 0 except for the bottom-right state (goal state)

- Actions: $A := \{L, R, U, D\}$

- 80% of the time actions lead to where they are supposed to.
  - The rest of the time (20%) they lead to a random adjacent state

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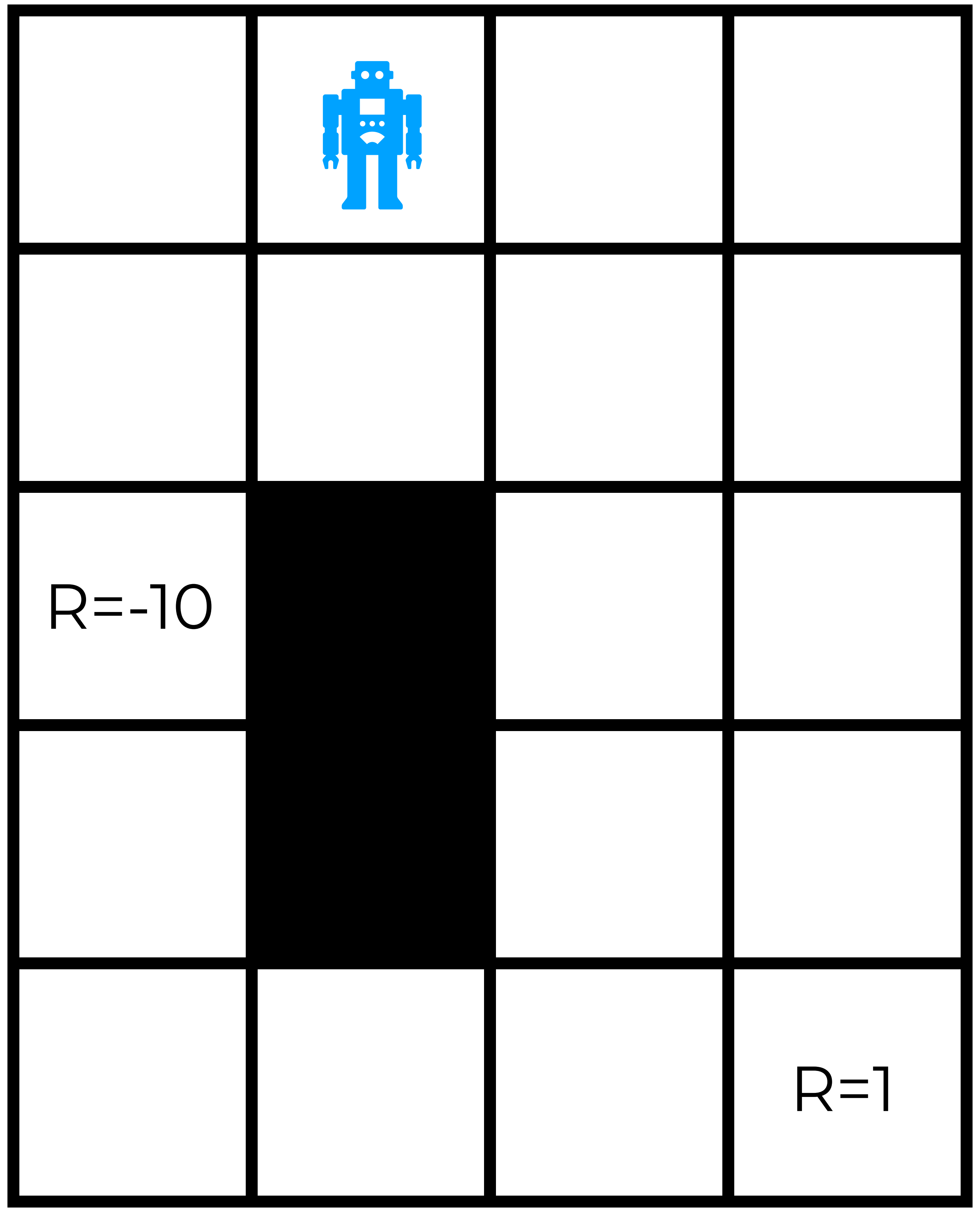
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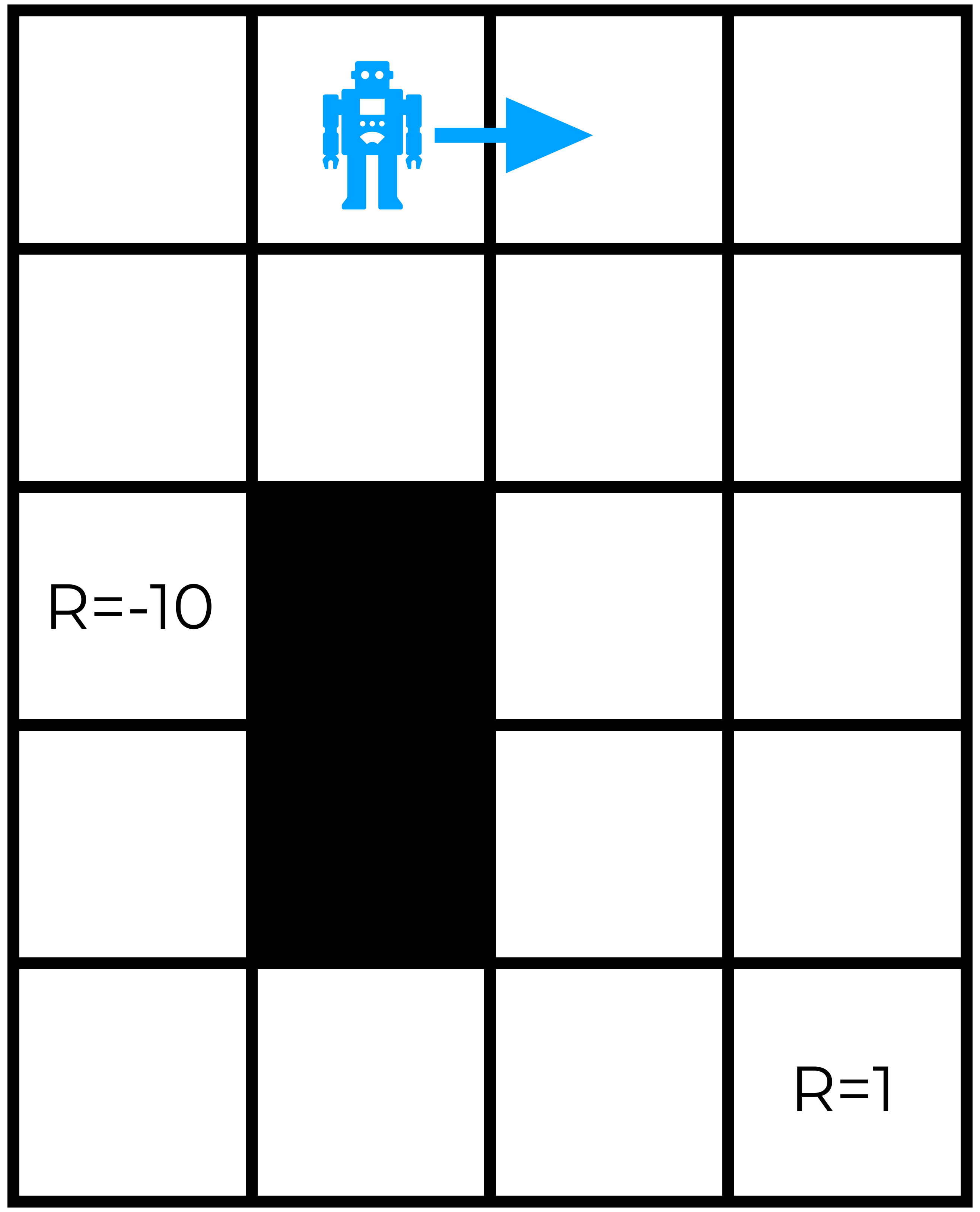
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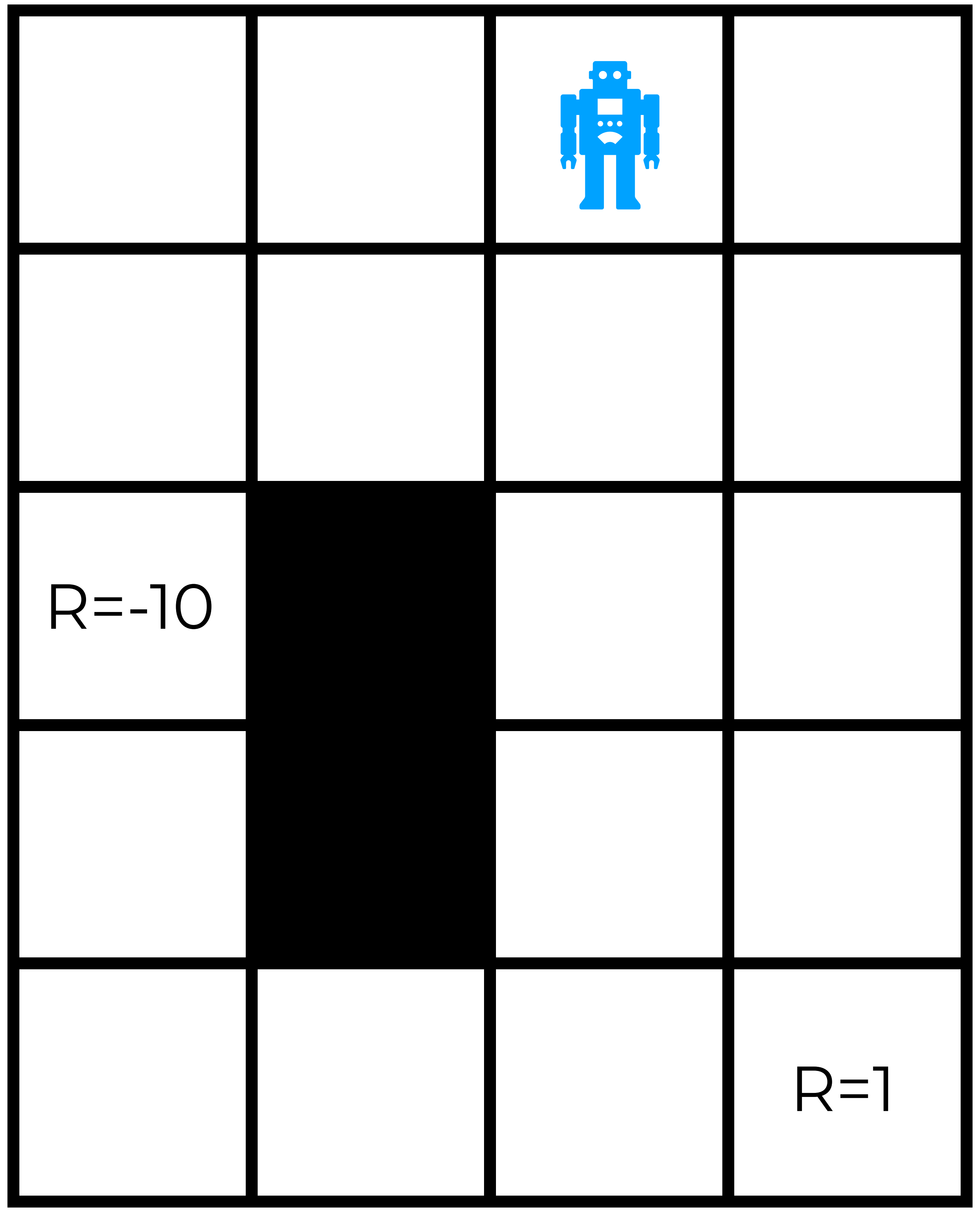
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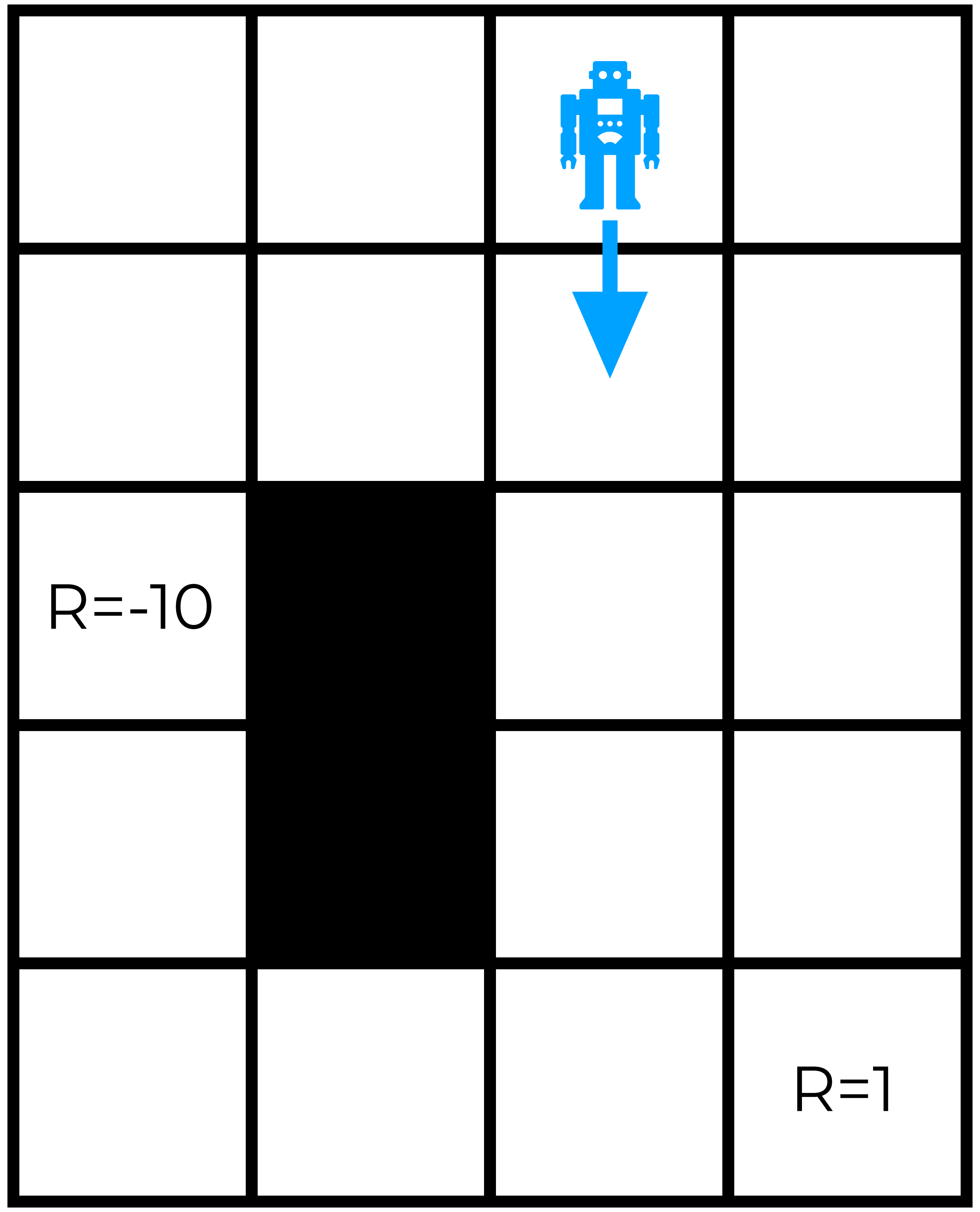
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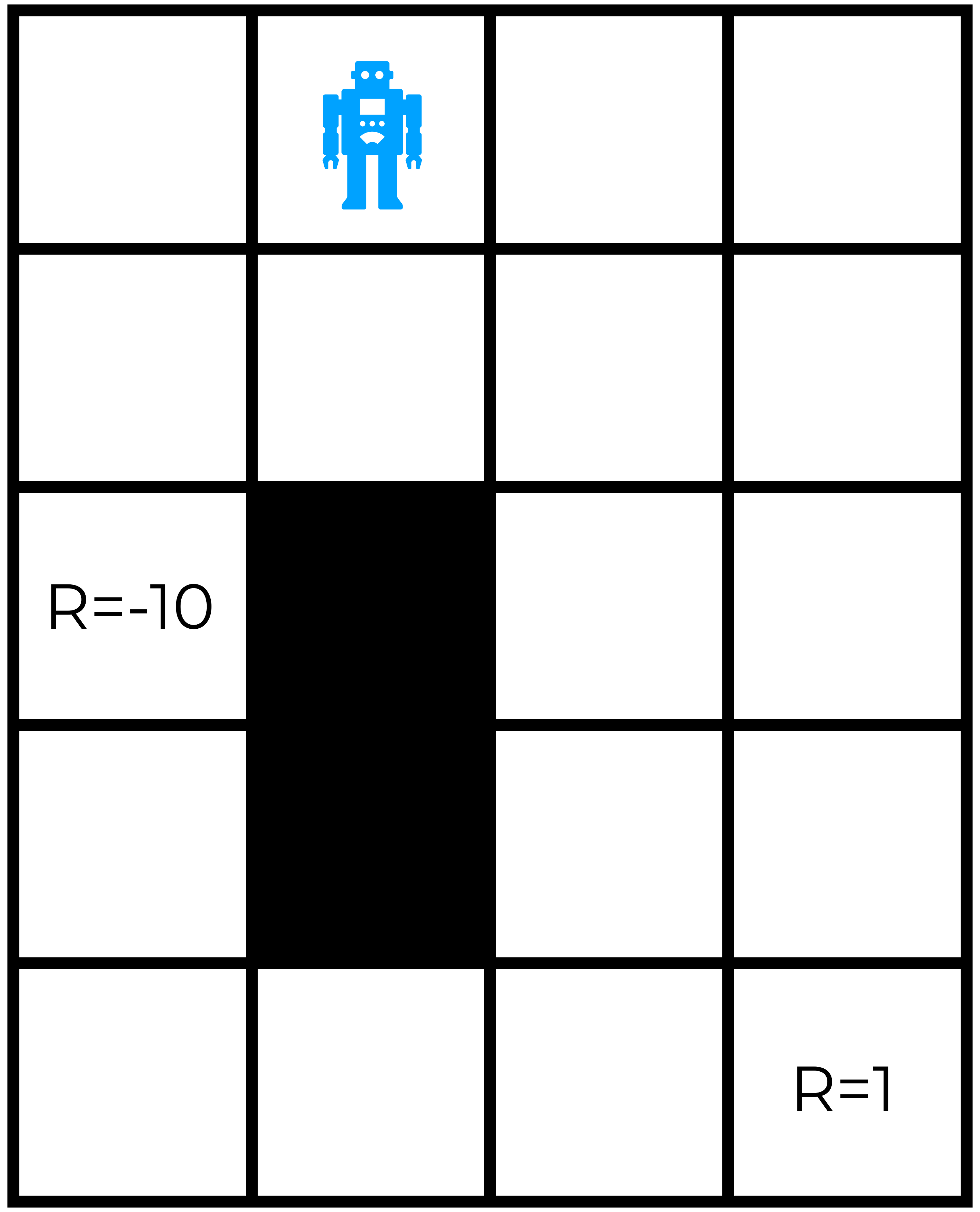
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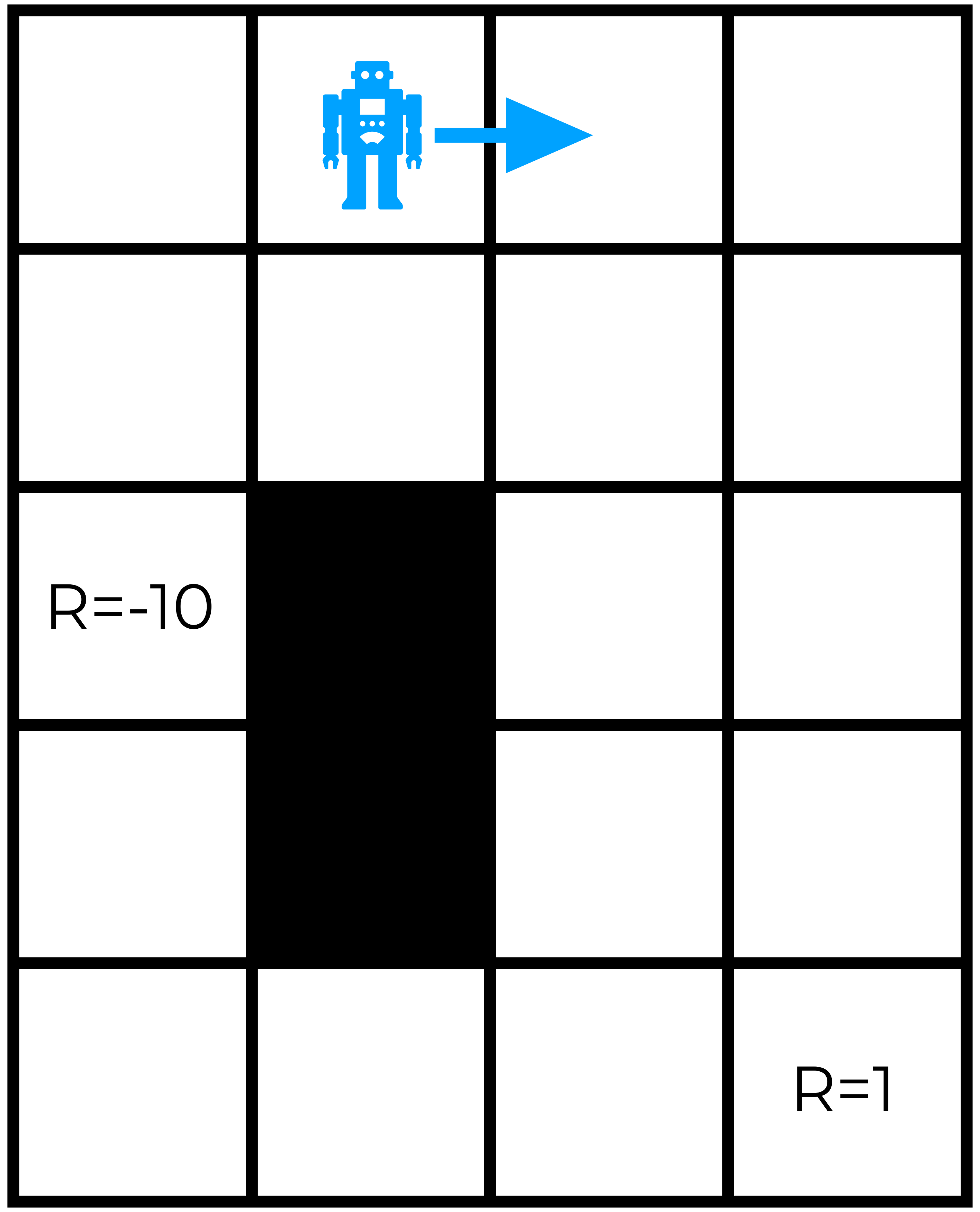
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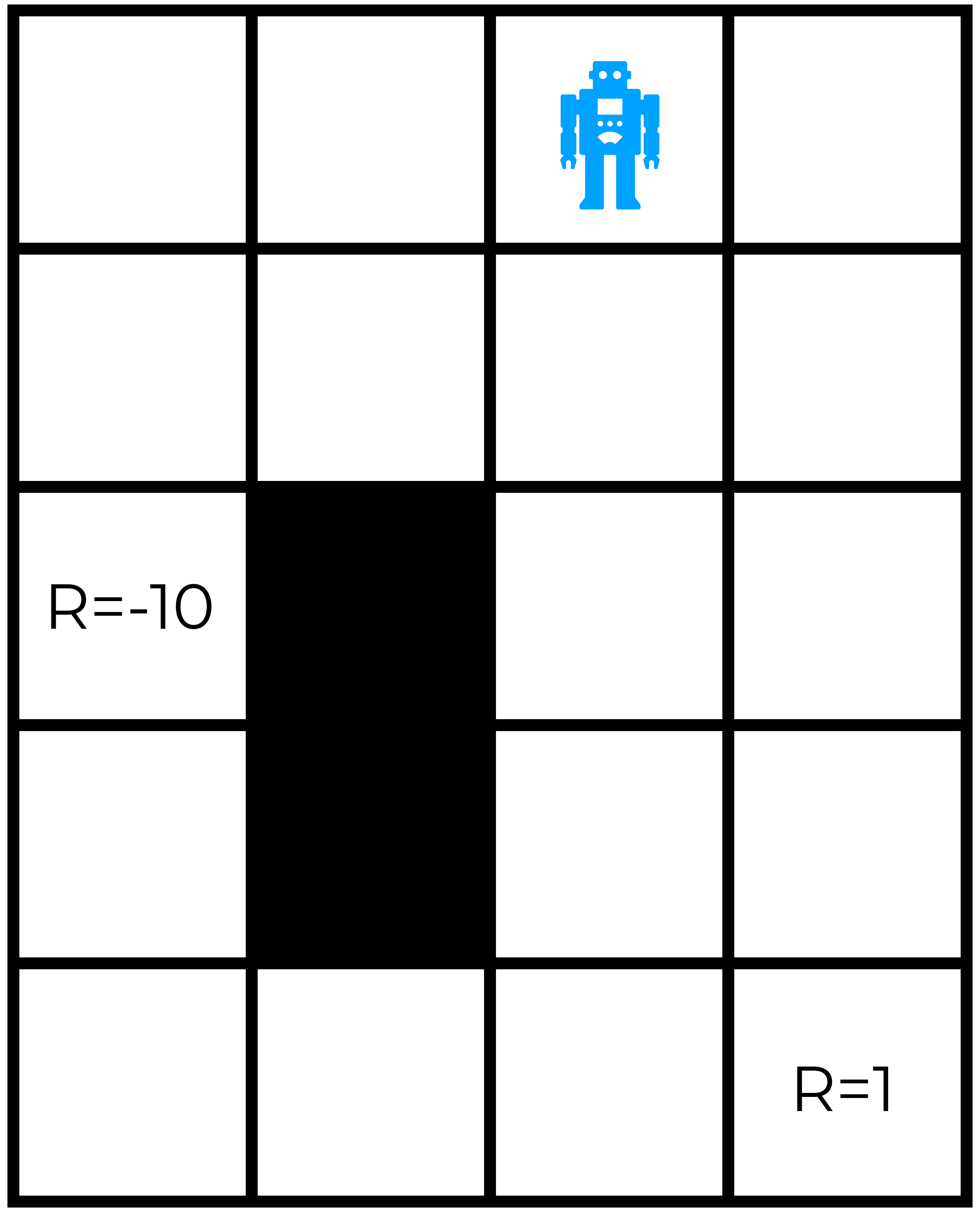
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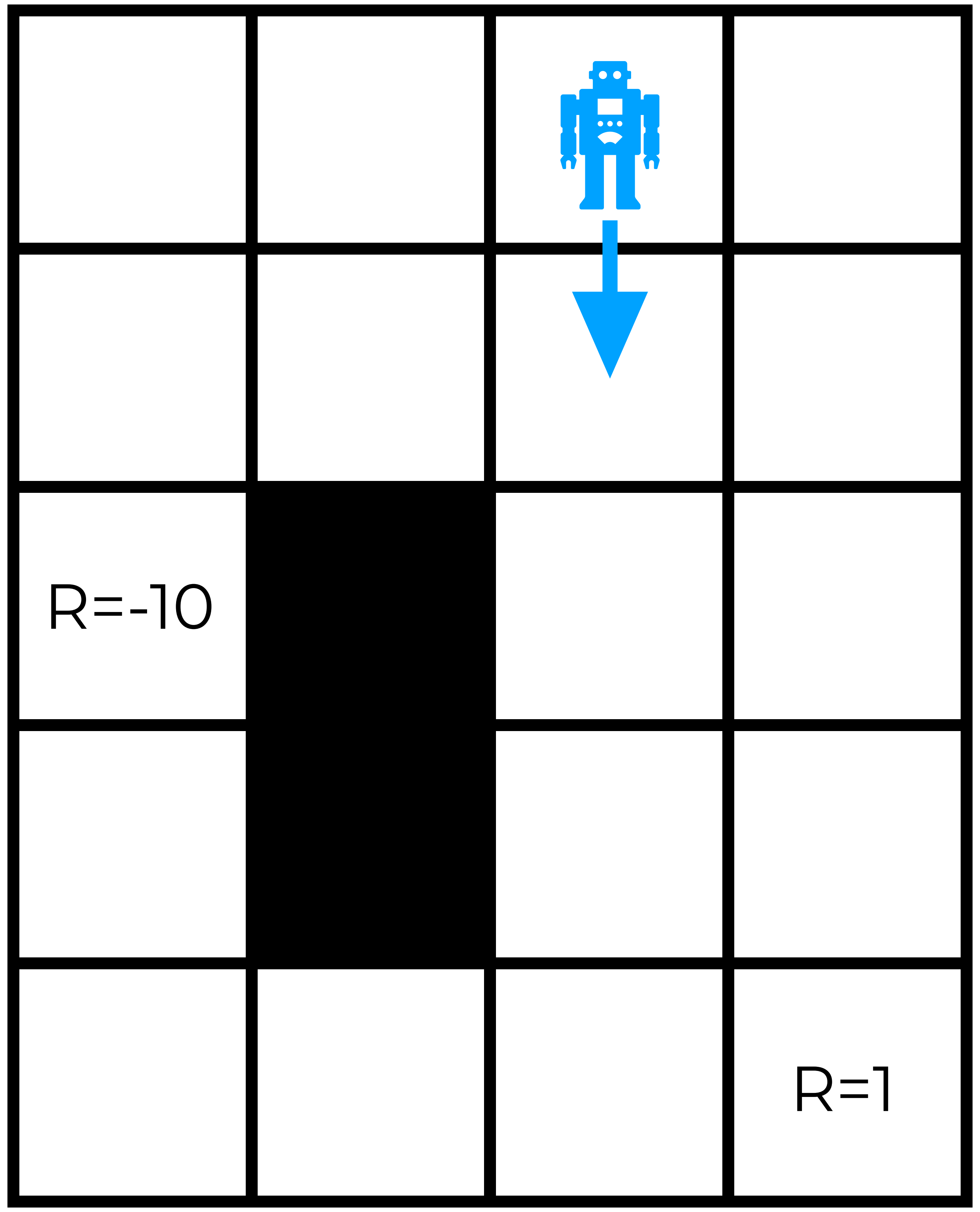
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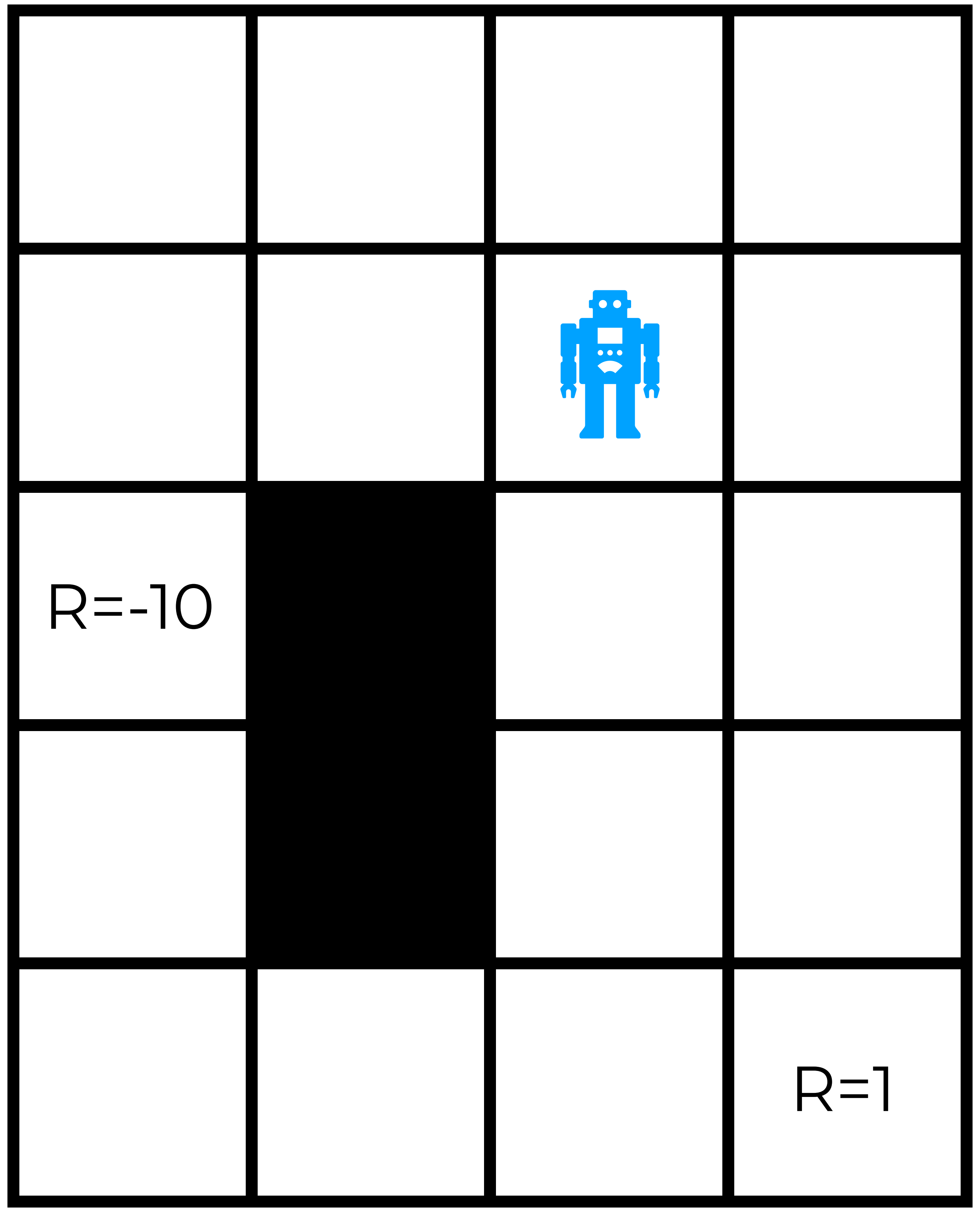
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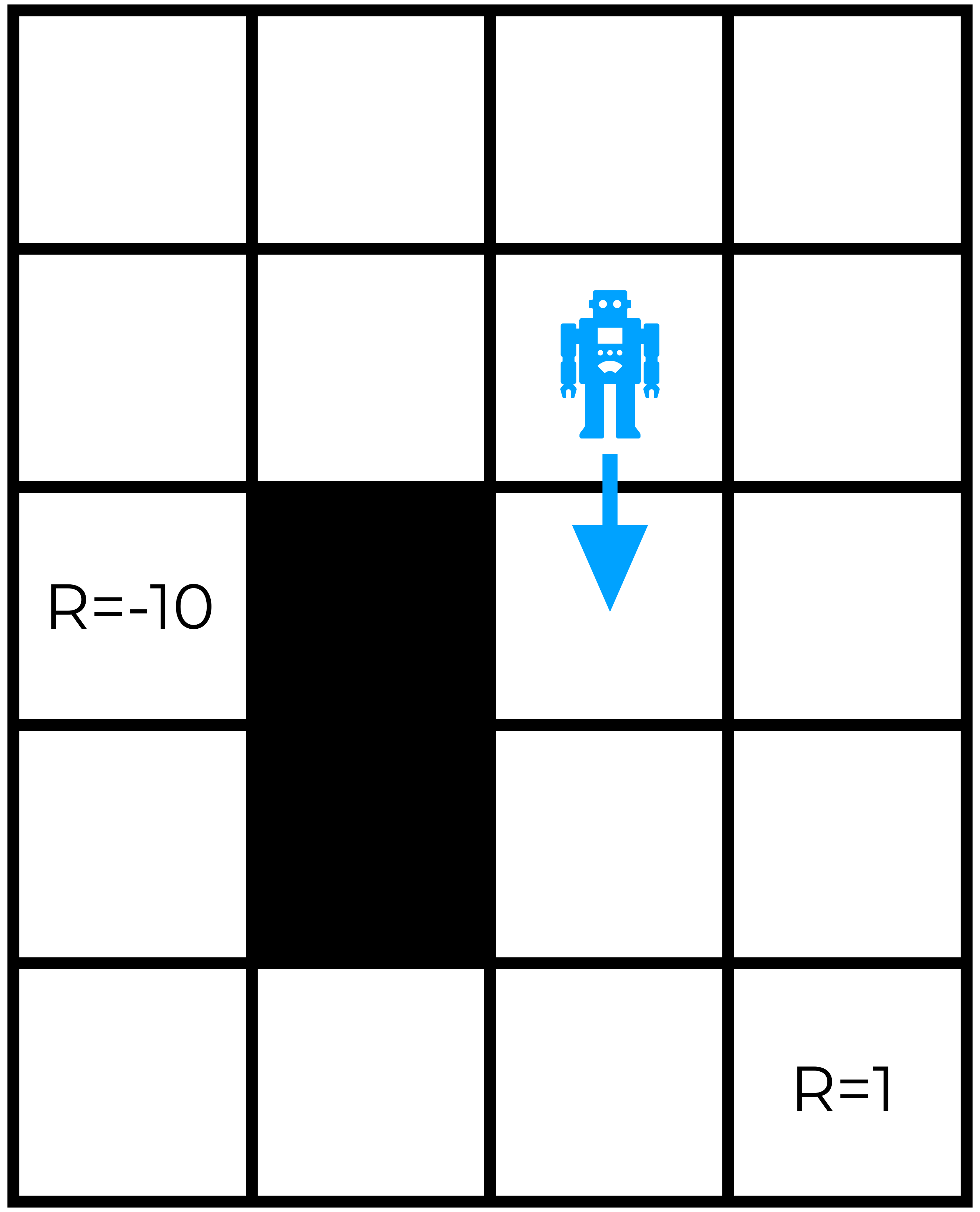
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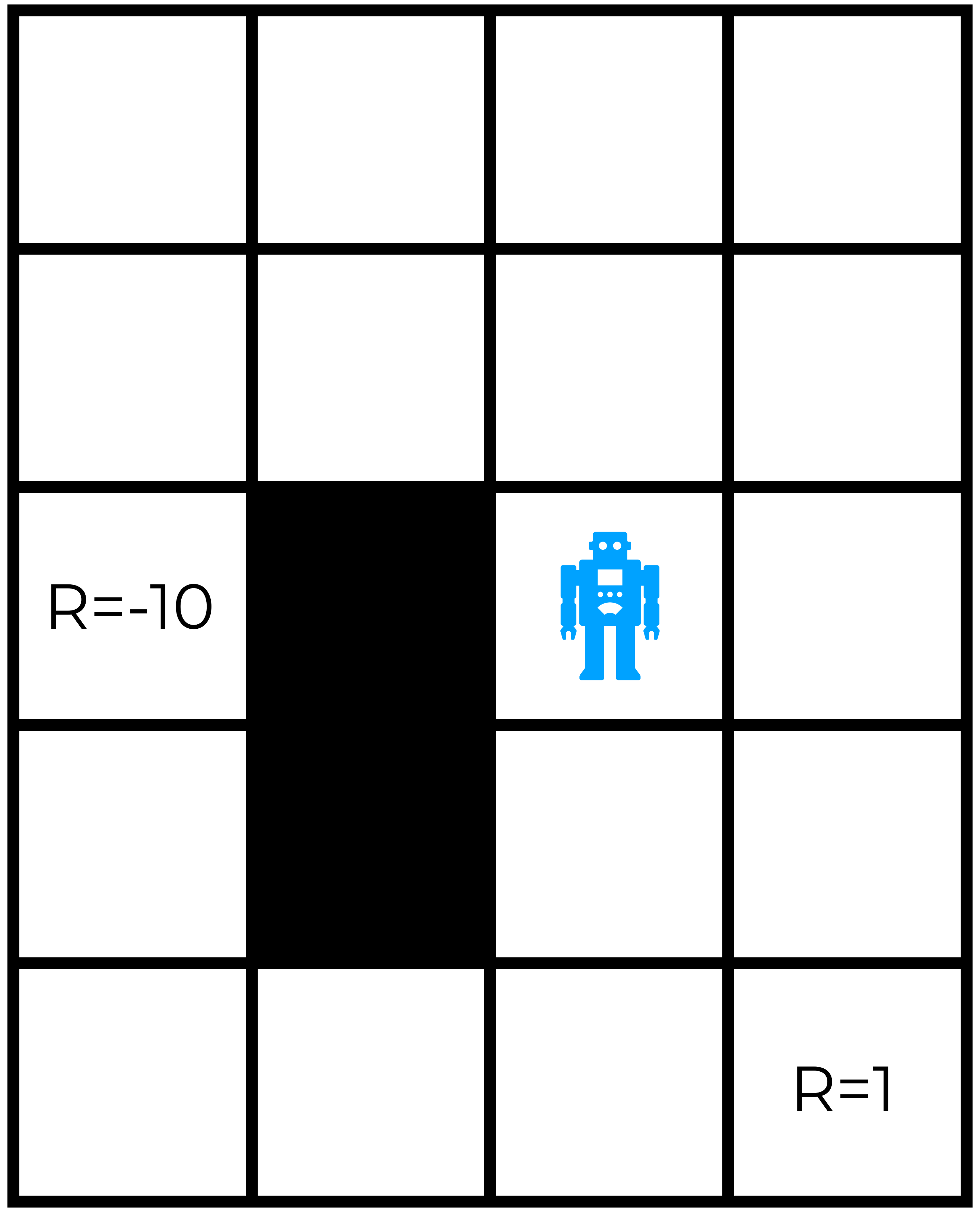
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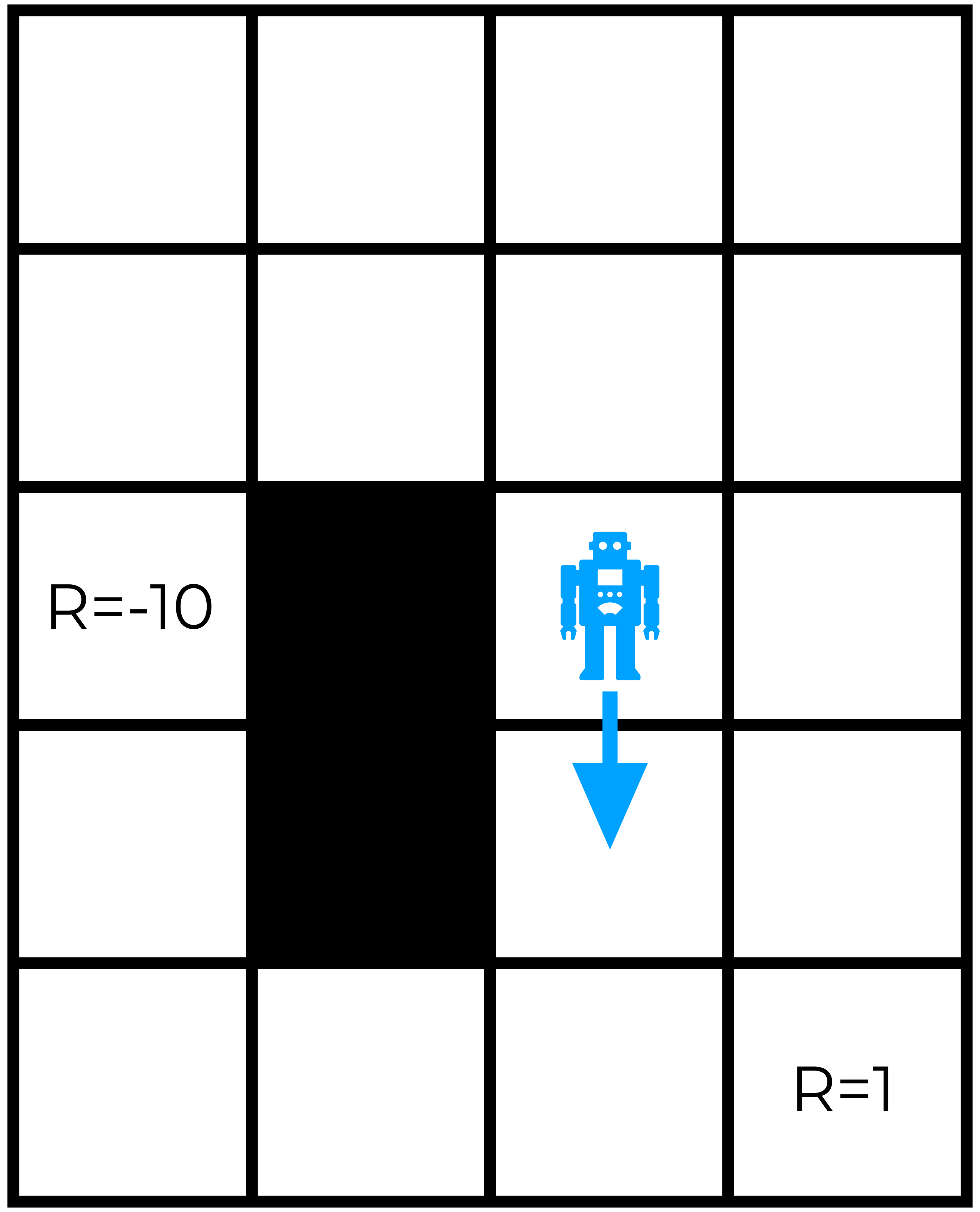
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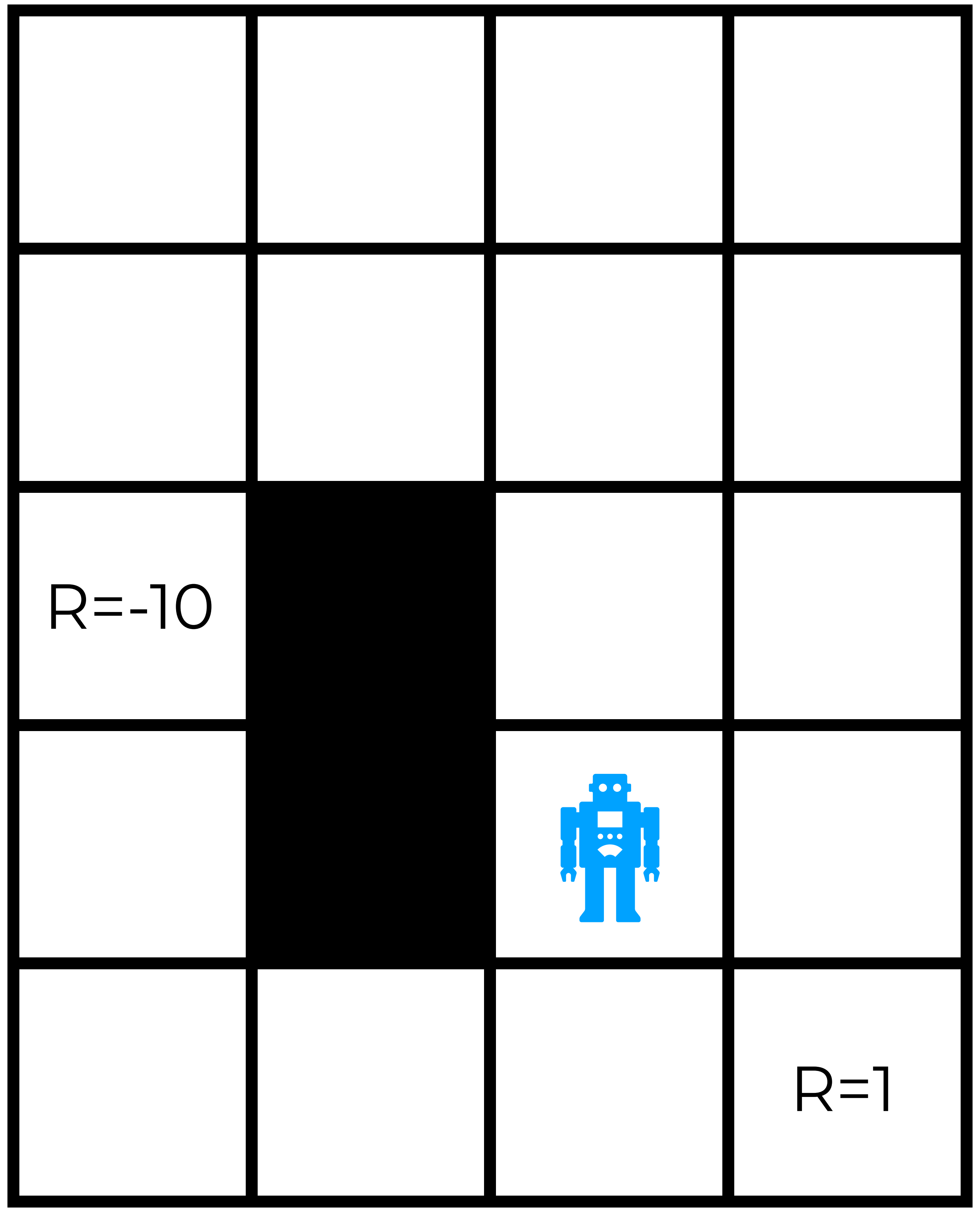
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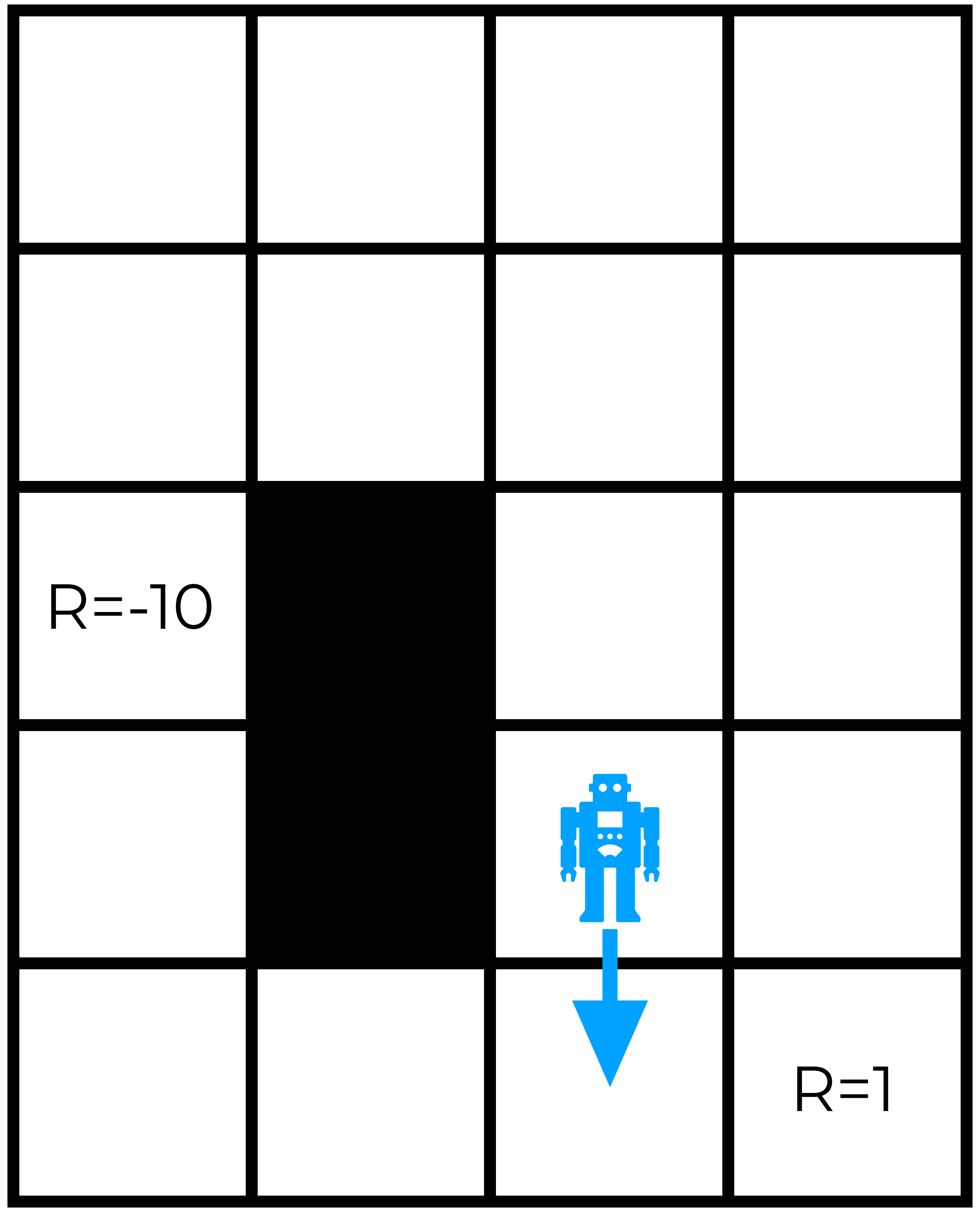
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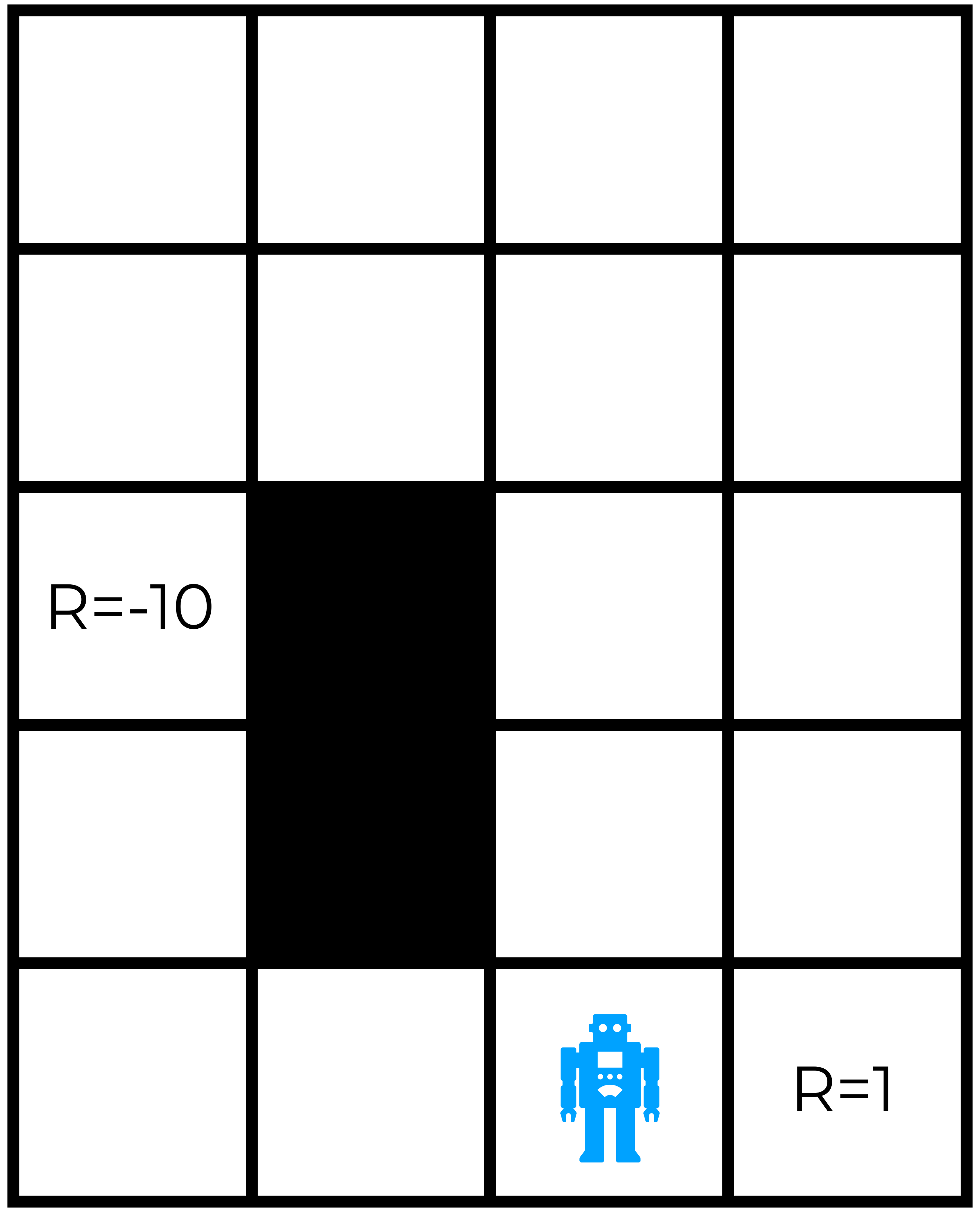
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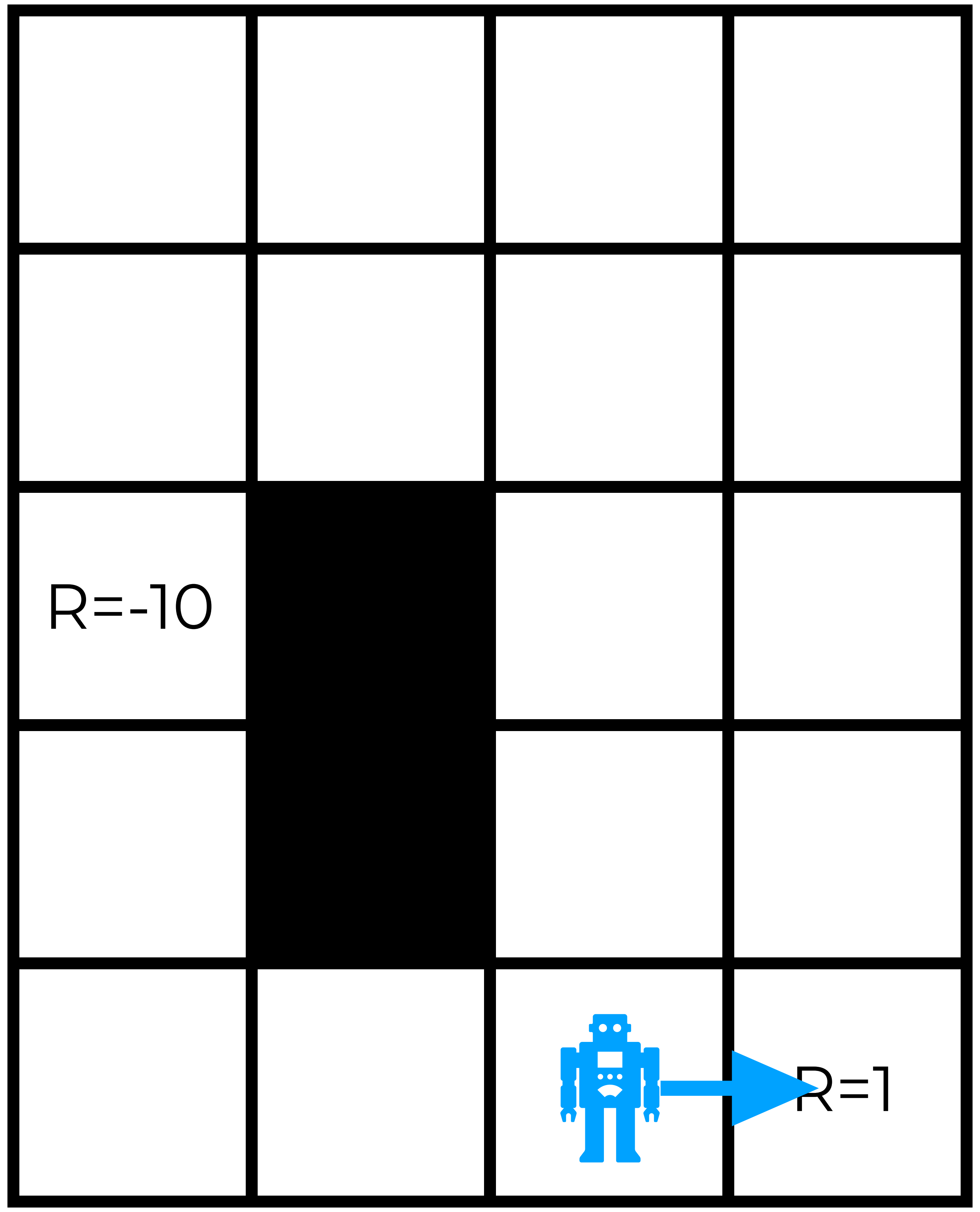
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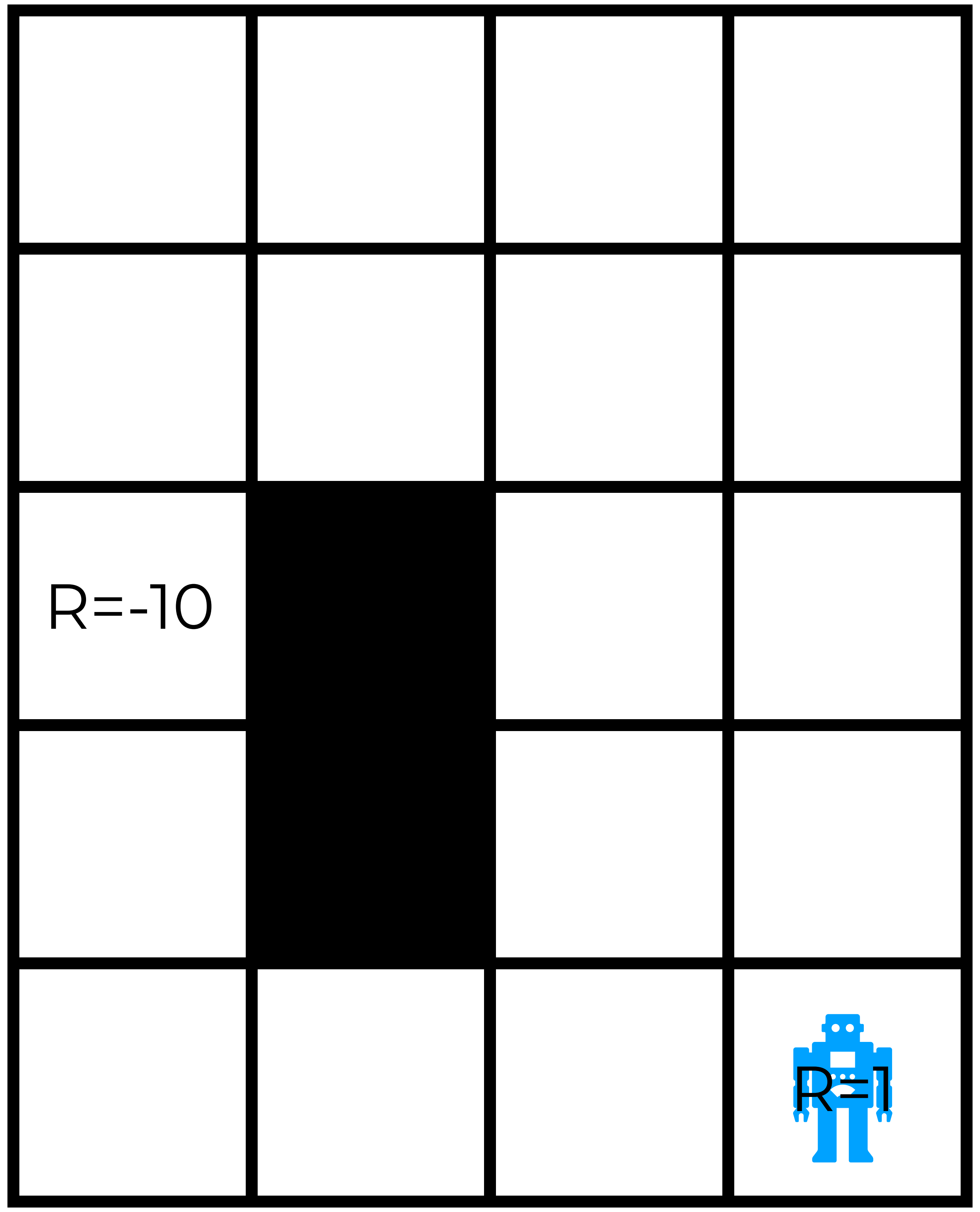
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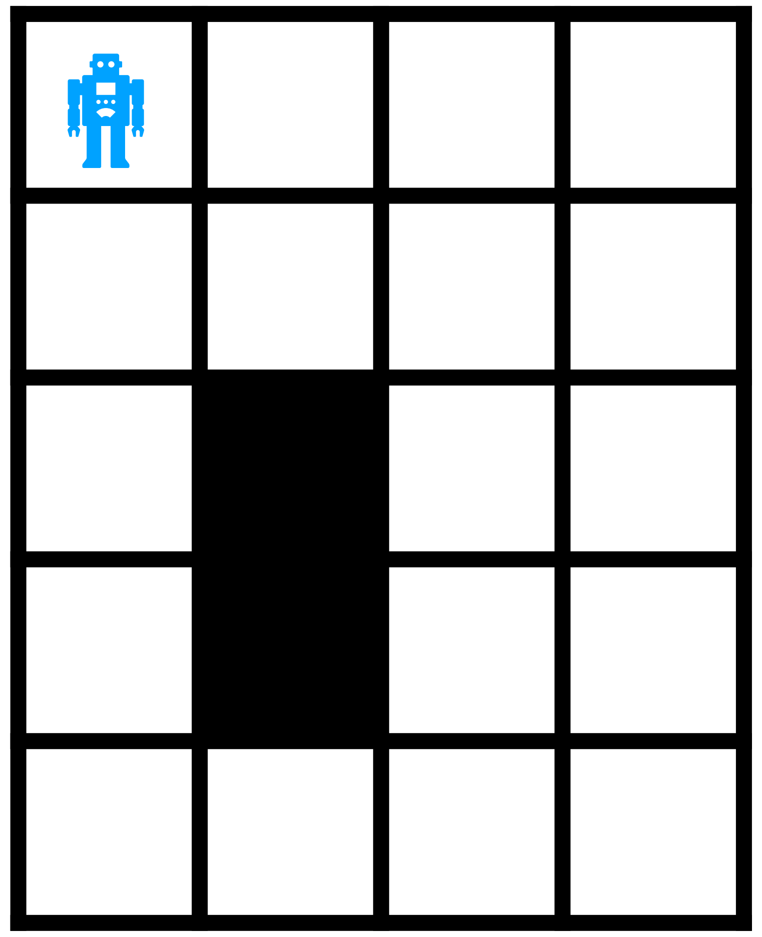
Markov Decision Process (MDP)

- Provide a framework for decision-making under uncertainty
  - Markov process with decisions and utilities
  - Assumes stationarity (i.e., transitions are fixed across time)



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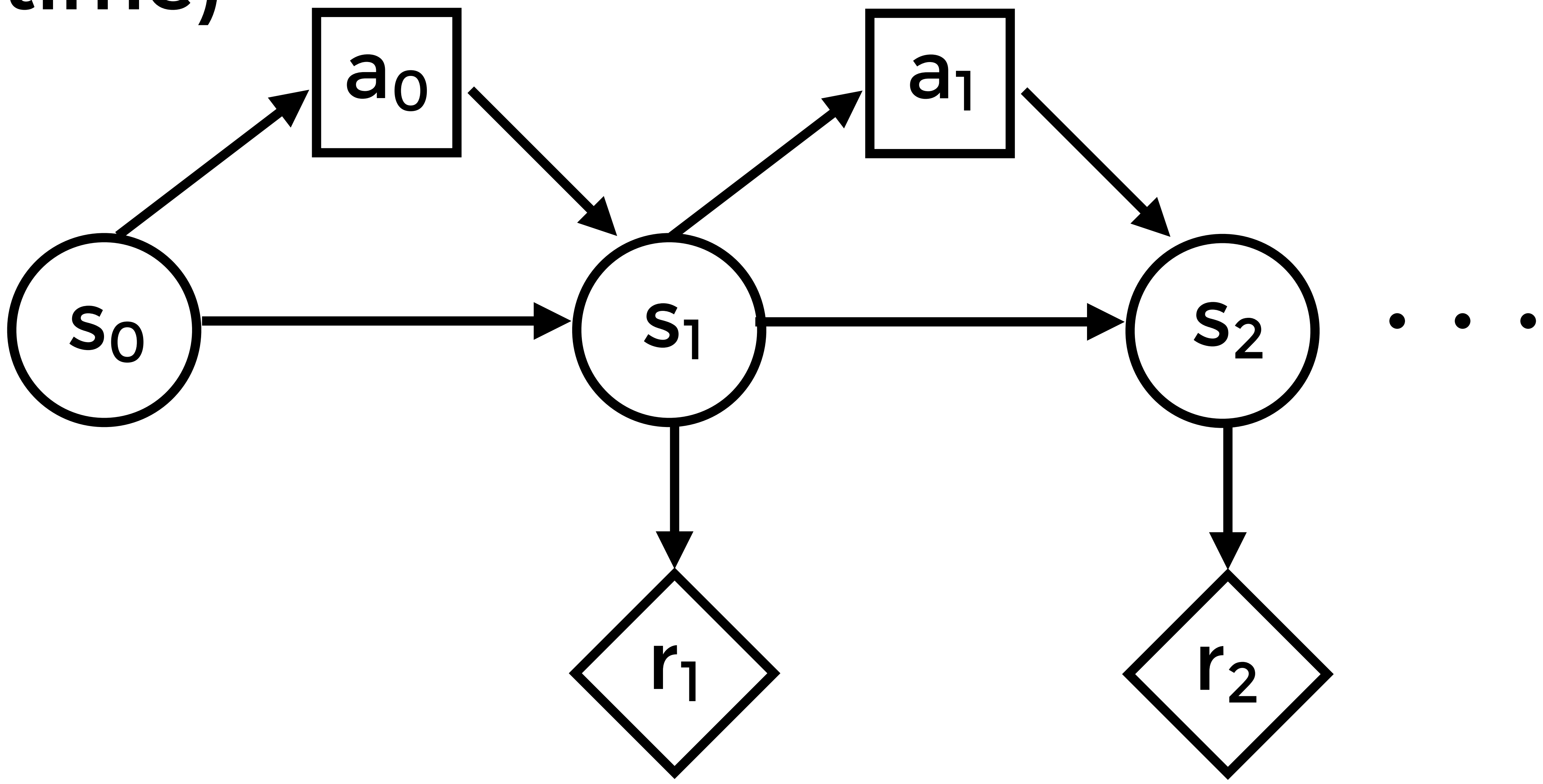
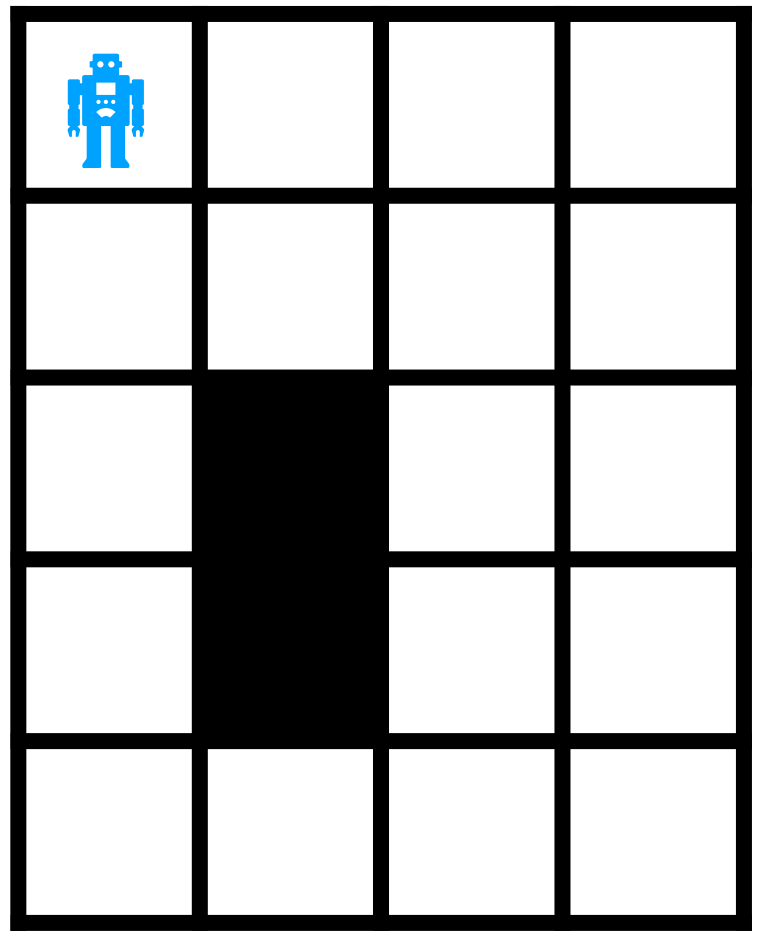
- Square nodes: decisions
- Circle nodes: States
- Diamond nodes: utility



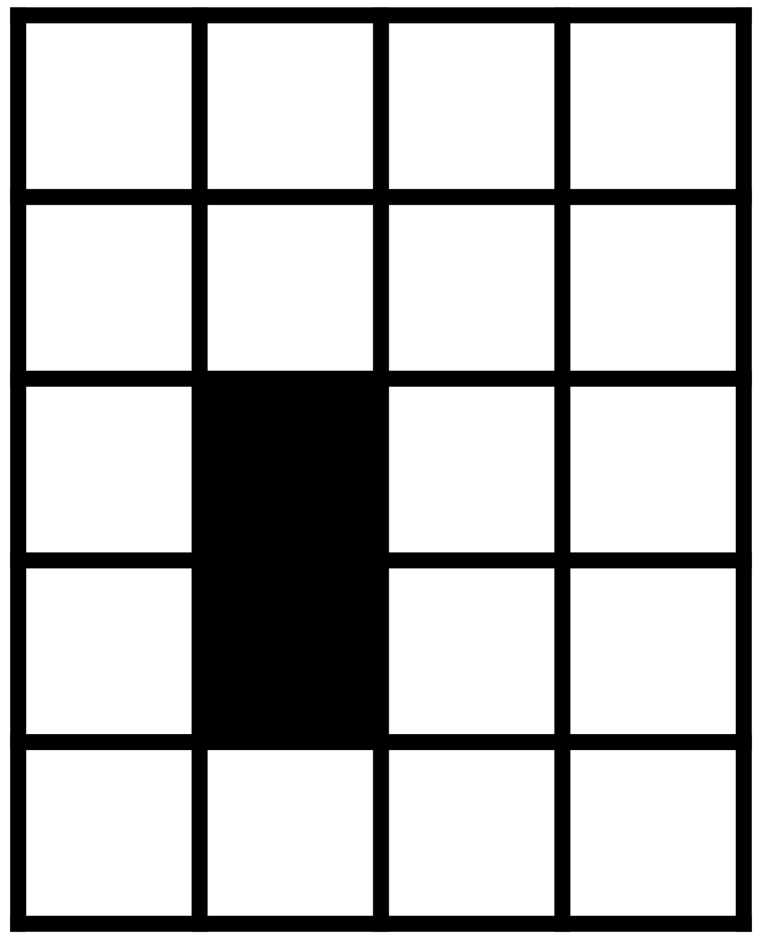


The objective of MDPs

Markov Decision Process (MDP)

$\langle \text{A}, \text{S}, \text{P}, \text{R}, \gamma \rangle$

- **A: set of actions**
- **P(S' | S,A): transition probabilities**
- **R(S): reward function**
- γ **: discount factor** $\in [0, 1]$



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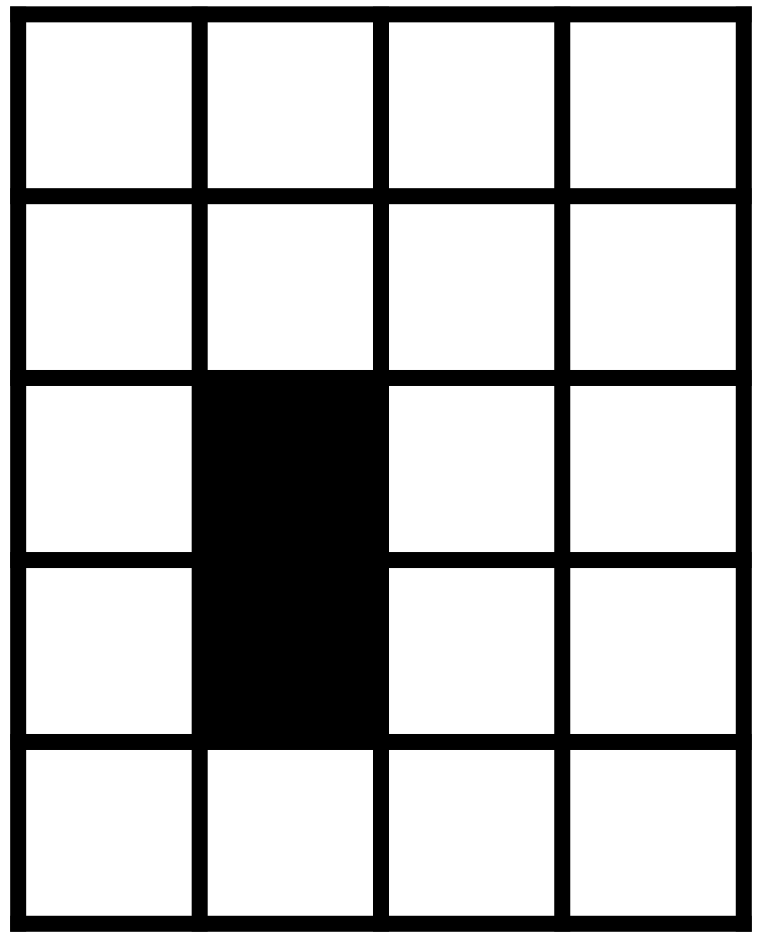

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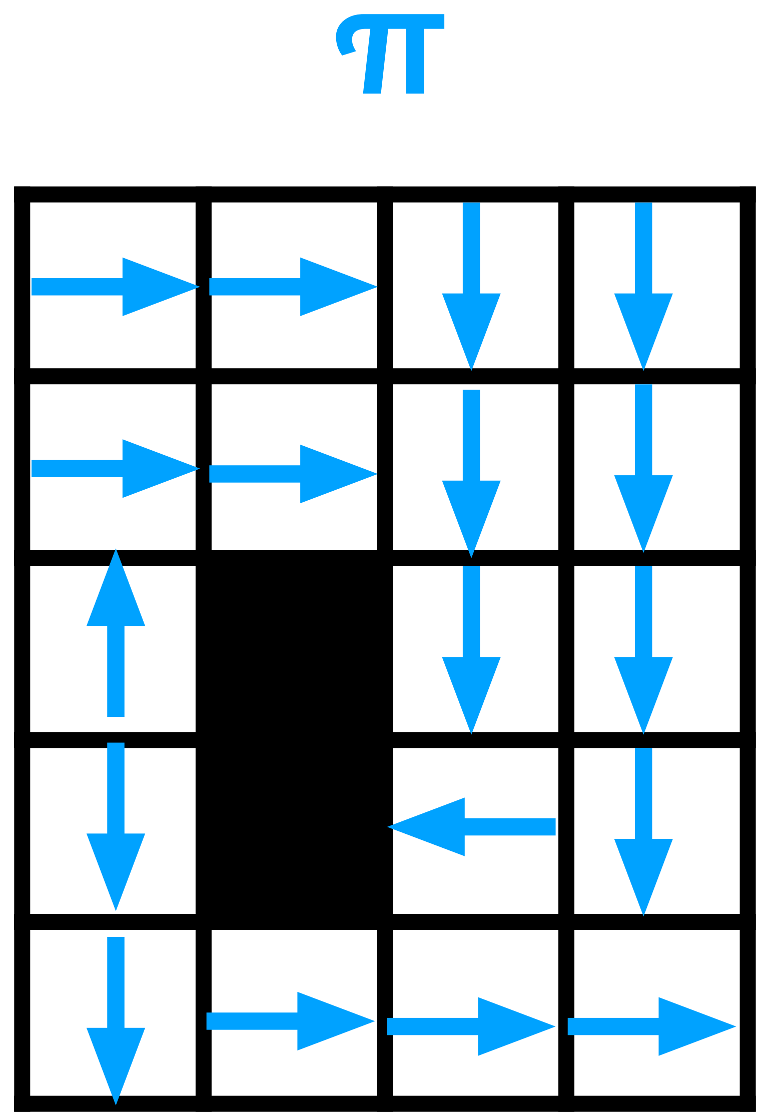

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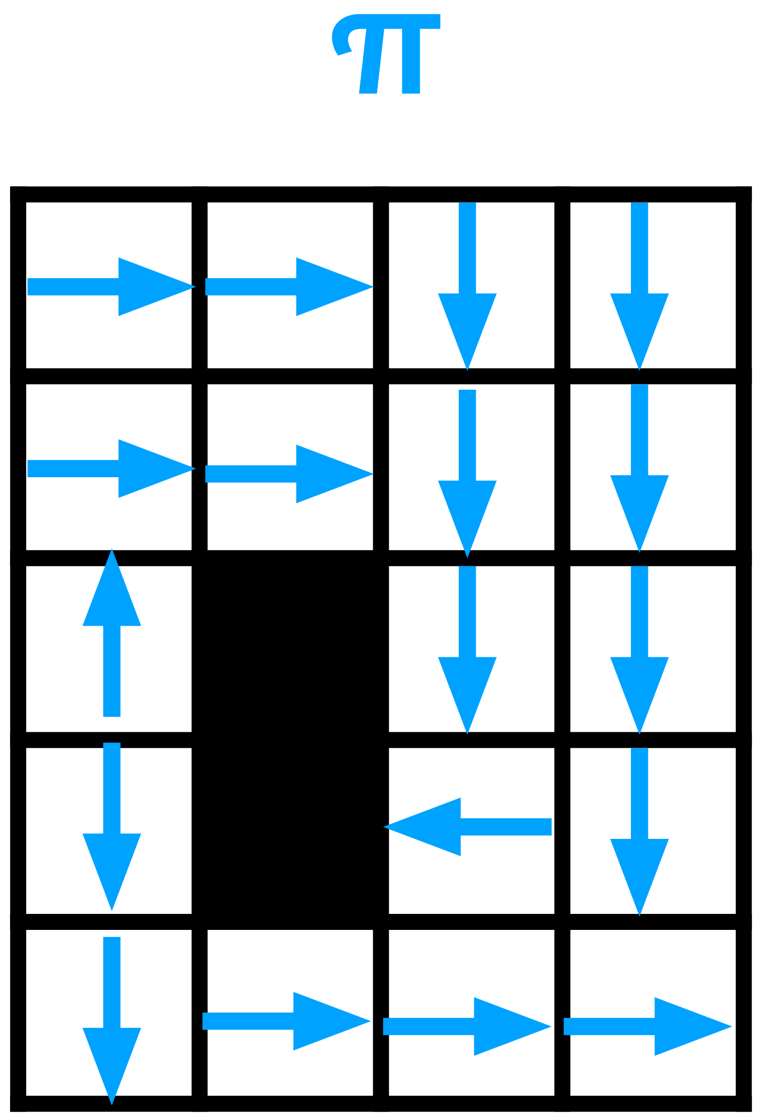
- A policy: π : S $\rightarrow$ A
- **Goal:** find the optimal policy



Optimal policy?

- Agent is trying to maximize its rewards (utility)
  - Utility simply assigns a real value to a state
  - Typically combine rewards with an additive function

$$\sum_t R(s_t)$$

Discounting (γ)

- **The sum of rewards could be infinite/unbounded**

$$\lim_{T \to \infty} \sum_{t}^{T} R(s_t)$$

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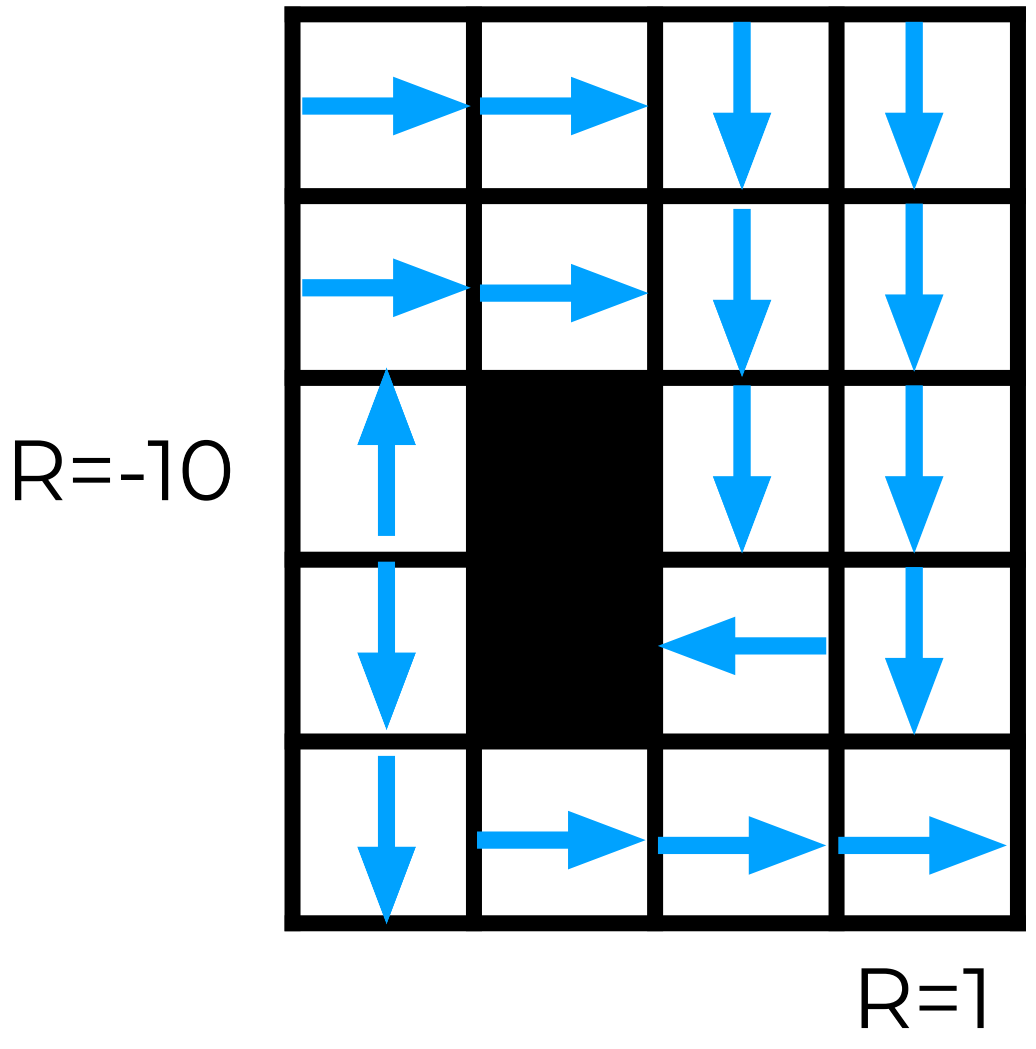
  - **Geometric series. Bounded by:** $\frac{R_{\max}}{1 - \gamma}$
  - **Intuition: would rather have rewards sooner**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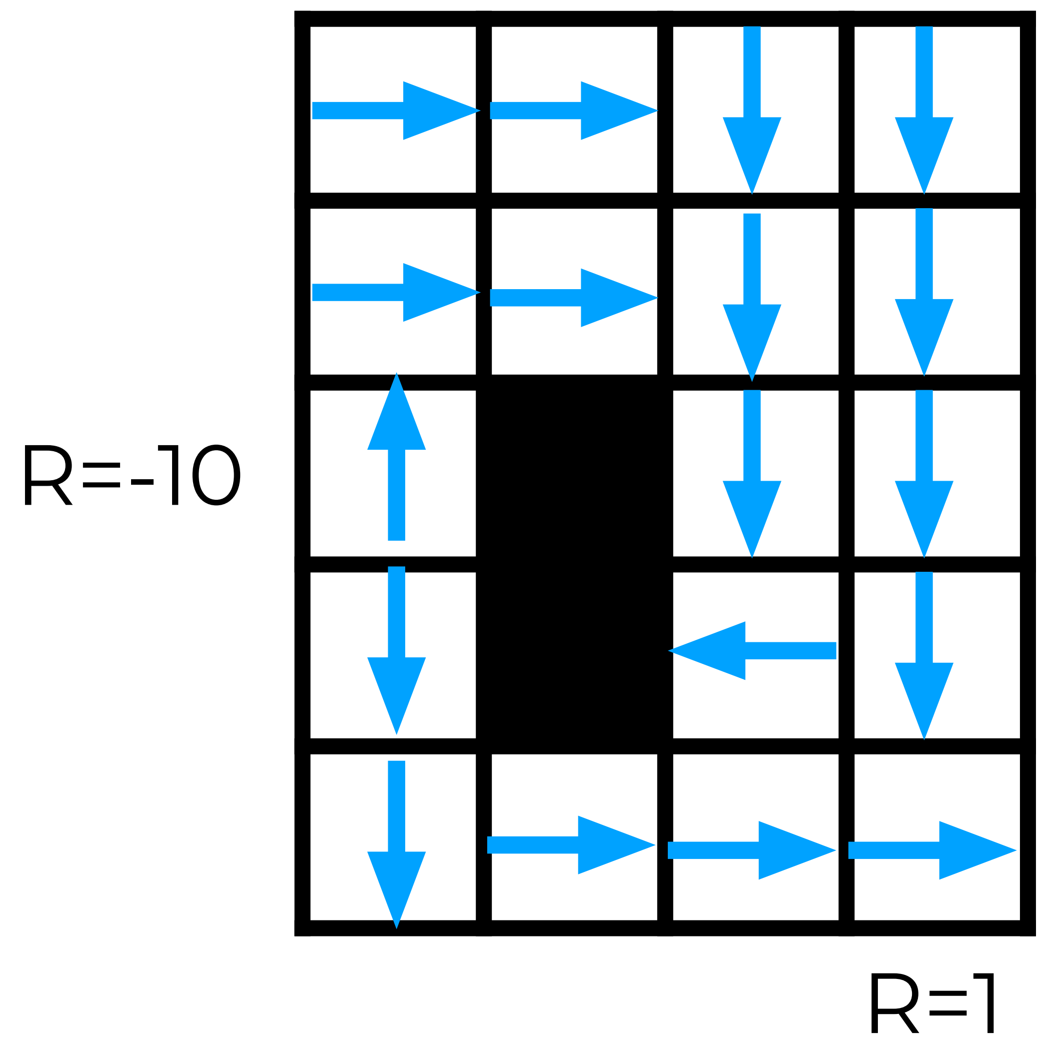

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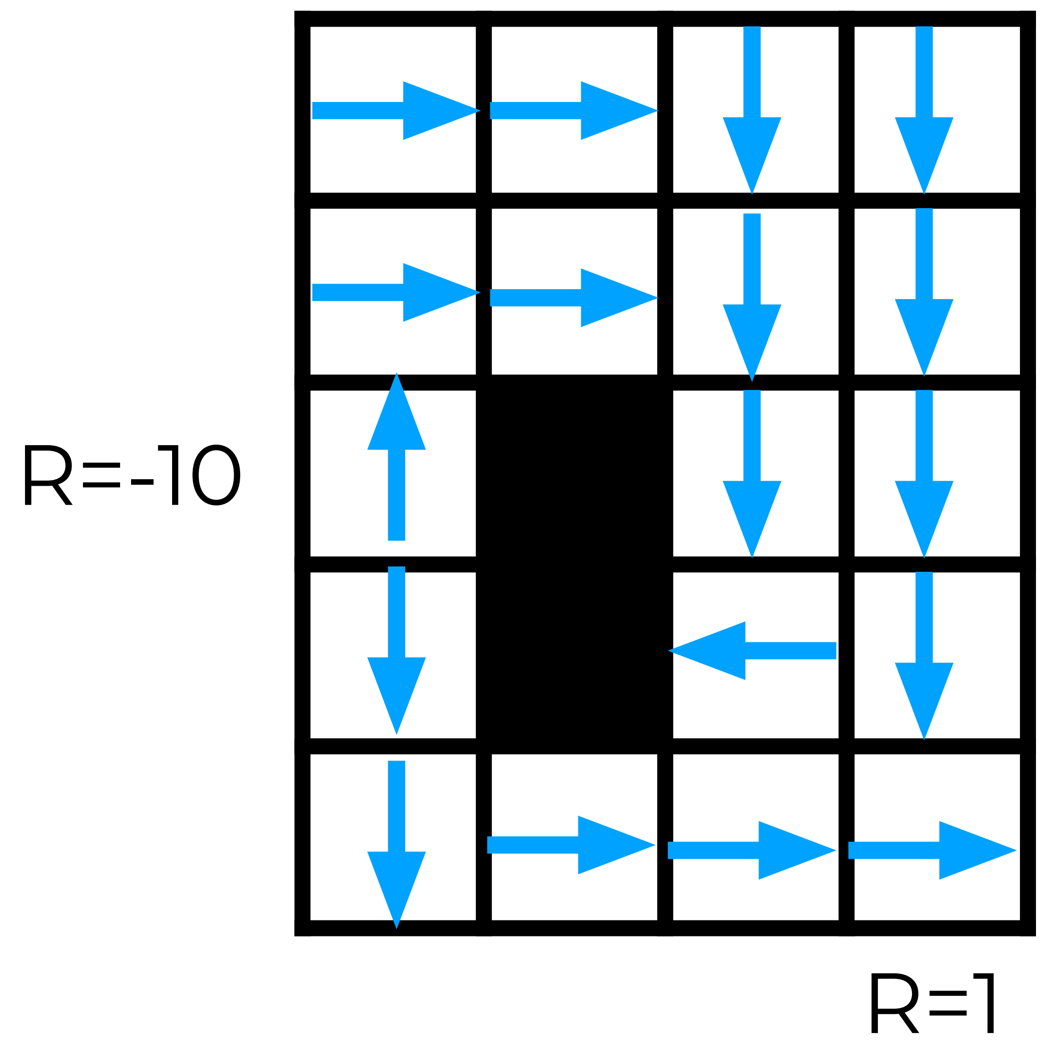

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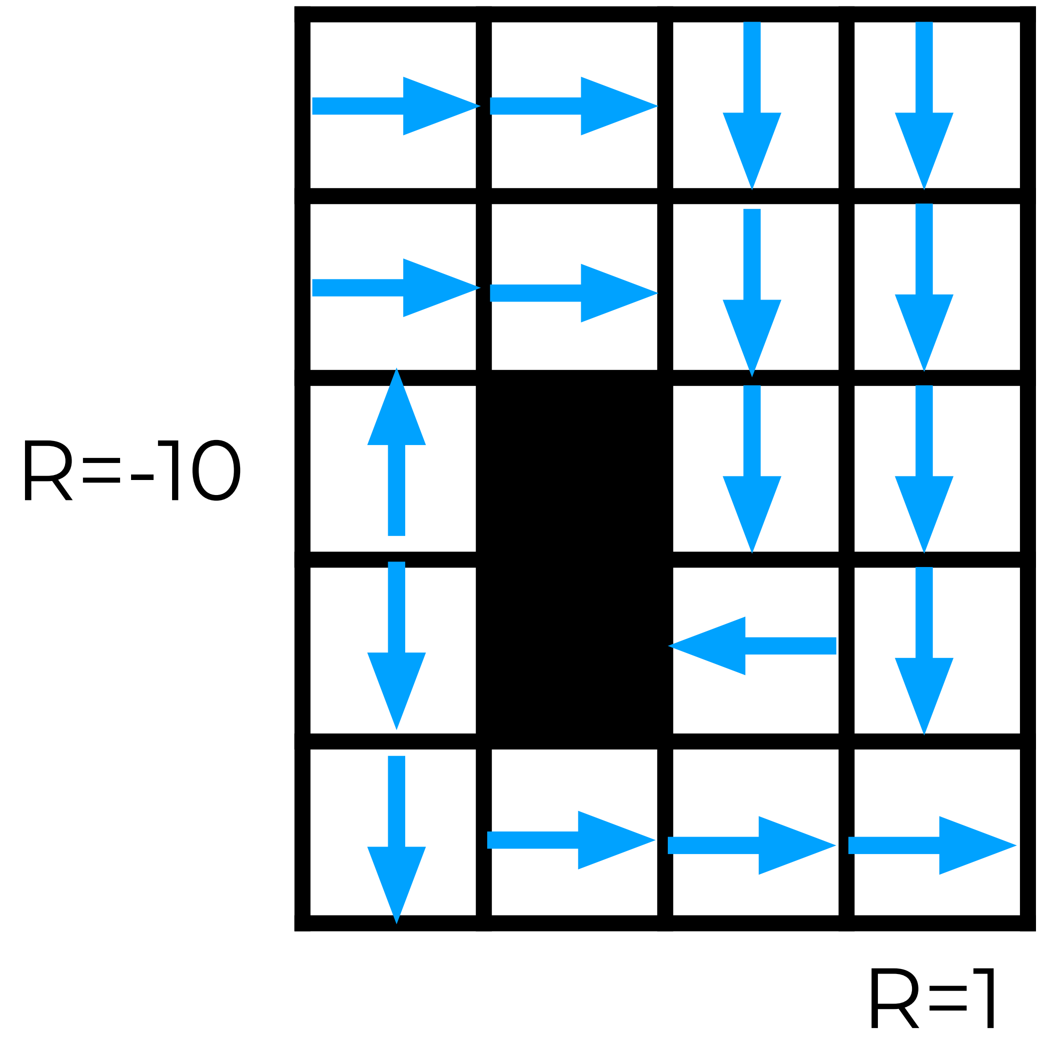

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Rewards are uncertain
- They depend on the transition probabilities

Maximize Expected Utility

- Maximize Expected Utility (MEU)
  - In short: optimal decision under uncertainty is the one with greatest expected utility
  - Variability comes from: environment uncertainty
- Justification for MEU: Rational agents must obey constraints which lead to optimizing expected utility

Solving an MDP

- **Find the optimal policy of an MDP**

$$\pi^*(s) \quad \forall s$$

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- The optimal policy is the one with highest expected utility: $EU(\pi^*) \geq EU(\pi) \quad \forall \pi$

Solving MDPs
(obtaining the optimal policy)

Solving an MDP

- Three well-known techniques:
  1. Value iteration
  2. Policy Iteration
  3. Linear Programming

Value Function

- $V(s_t)$: The value of being in state s at time t

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$$V(s_t) := \text{expected sum of rewards of being in s}$$

Finite horizon

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Finite horizon

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$$V(s_{T-1}) = \max_{a_{T-1}} \left\{ R(s_{T-1}) + \gamma \sum_{s_T} P(s_T \mid s_{T-1}, a_{T-1}) R(s_T) \right\}$$

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Bellman equation

- **Value of state s**

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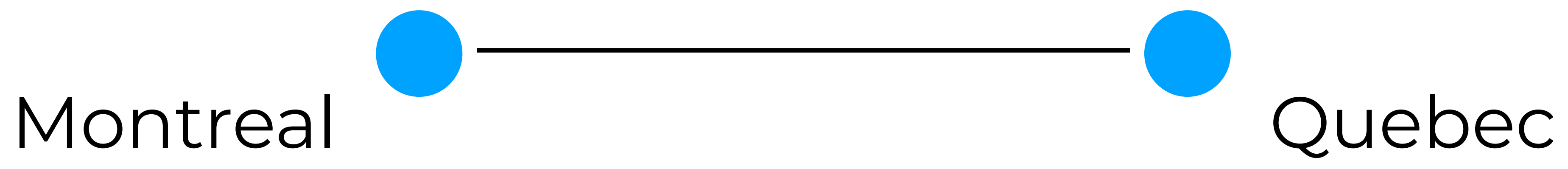
- Recursive equations
- The value of a state only depends on the state's reward and the neighbours' value
- This is also known as a dynamic programming equation

Dynamic Programming (in 1 slide)

- **Solution technique that decomposes a problem into a set of subproblems**
  - **The solution to each subproblem is part of the solution of the original problem**
  - **E.g.,**



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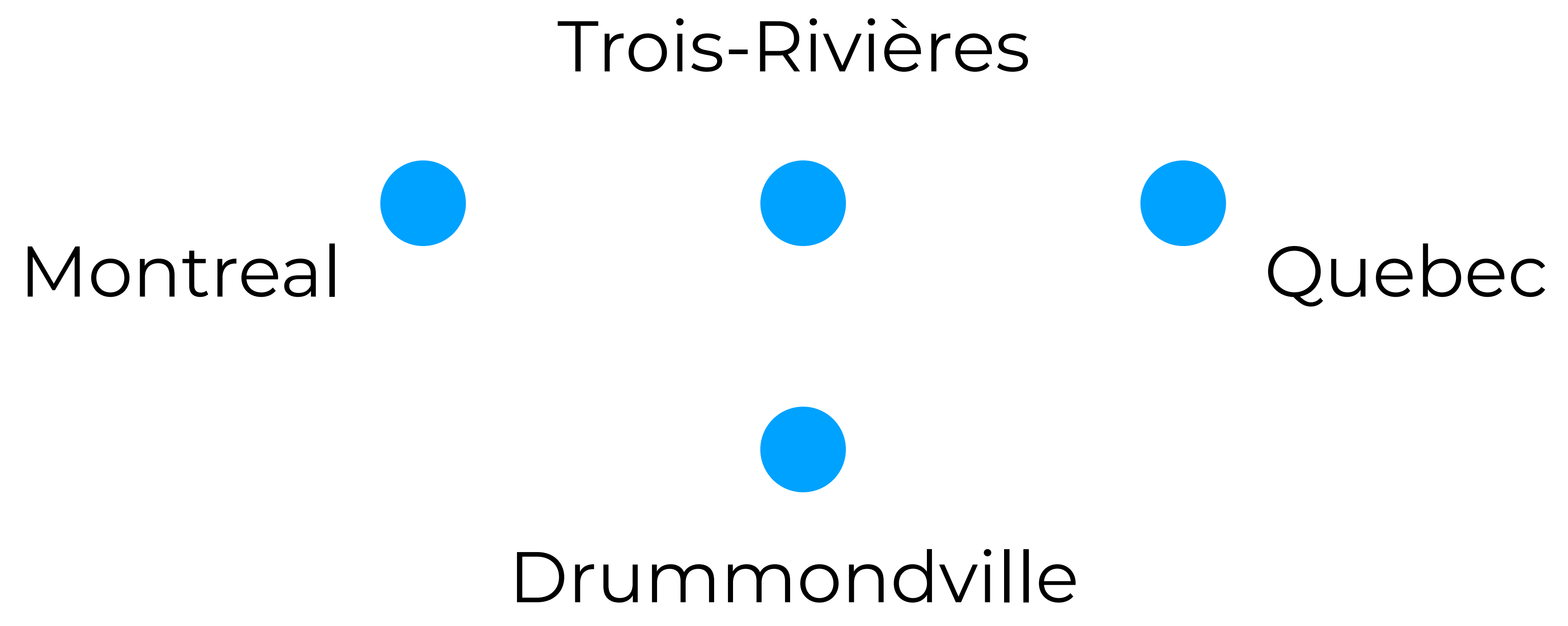

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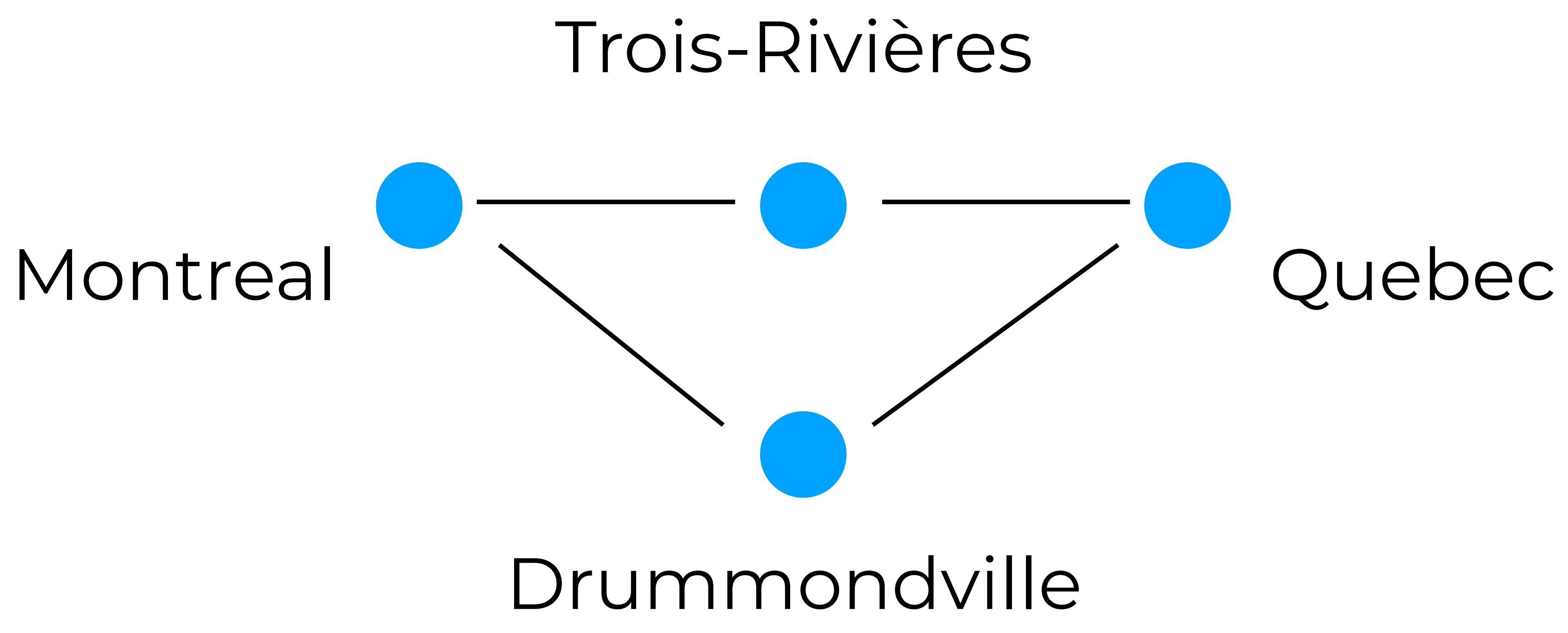

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- For i=1,2,3,...
  - For s=1,...,S

$$\mathbf{V}(\mathbf{s}) = \max_{\mathbf{a}} \left\{ \mathbf{R}(\boldsymbol{s}) + \gamma \sum_{\mathbf{s}'} \mathbf{P}(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}) \mathbf{V}(\mathbf{s}') \right\}$$

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- For i=1,2,3,...
  - For s=1,...,S

$$\mathbf{V}(\mathbf{s}) = \max_{\mathbf{a}} \left\{ \mathbf{R}(\boldsymbol{s}) + \gamma \sum_{\mathbf{s}'} \mathbf{P}(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}) \mathbf{V}(\mathbf{s}') \right\}$$

- The policy is implicit
  - Once converged: $\boldsymbol{\pi}^*(\mathbf{s}) = \arg\max_{\mathbf{a}} \left\{ \mathbf{R}(\mathbf{s}) + \gamma \sum_{\mathbf{s}'} \mathbf{P}(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}) \mathbf{V}^*(\mathbf{s}') \right\} \forall \mathbf{s}$

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2. Update the current policy

$$\pi'(s) = \arg\max_{a} \left\{ R(s) + \gamma \sum_{s'} P(s' \mid s, a) V^{\pi}(s') \right\} \quad \forall s$$

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Start with any (e.g., random) policy π

Iterate until convergence:

Policy Evaluation

1. Given current policy get the value of each state

$$V^{\pi}(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) V^{\pi}(s') \qquad \forall s$$

Policy Update

2. Update the current policy

$$\pi'(s) = \arg\max_{a} \left\{ R(s) + \gamma \sum_{s'} P(s' \mid s, a) V^{\pi}(s') \right\} \quad \forall s$$

PI vs. VI

- **Value iteration is faster per iteration**
- **Policy iteration converges in fewer iterations**

- **Some of these slides were adapted from Pascal Poupart's slides (CS686 U.Waterloo)**