Closure Operators for Order Structures*

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Abstract. We argue that closure operators are fundamental tools for the study of relationships between order structures and their sequence representations. We also propose and analyse a closure operator for interval order structures.

1 Introduction

While the two major models of concurrency, interleaving abstraction ([2, 22]) and partially ordered causality ([5, 15, 23]), have been very successful, they have some limitations. Neither of them can model the "not later than" relationship effectively, which causes problems with specifying priorities, error recovery, time testing, inhibitor nets, etc. (see for instance [4, 9, 12, 16–18]). A solution, proposed independently (in this order) in [19, 8] and [10], suggests modeling concurrent behaviours by a triple $(X, \prec, \sqsubset)$, where X is the set of event occurrences, and $\prec$ and $\sqsubset$ are binary relations on X. The relation $\prec$ is "causality" (i.e. an abstraction of the "earlier than" relationship), and $\sqsubset$ is "weak causality" (an abstraction of the "not later than" relationship). For this model, the following two kinds of relational structures are of special importance: *stratified order structures* (*so-structures*) and *interval order structures* (*io-structures*). The former structures can fully model concurrent behaviours when system executions (operational semantics) are described in terms of stratified orders, while the latter structures can fully model concurrent behaviours when system executions are described in terms of interval orders [9, 13]. It was argued in [11] (and also implicitly in 1914 Wiener's paper [26]) that any execution that can be observed by a single observer must be an interval order. Thus, io-structures provide a very general model of concurrency. However, the theory of io-structures is far less developed than the simpler theory of so-structures.

When dealing with partial orders, many constructions use the fundamental notion of *transitive closure* of relations. The analogue of transitive closure for so-structures, called $\diamondsuit$-*closure*, has been proposed in [12] and successfully used in [12, 16–18] and others. However, a similar concept for io-structures has not yet been proposed. In this paper we introduce the concept of $\blacklozenge$-*closure* for io-structures and show that it has the same kind of properties as transitive closure and $\diamondsuit$-closure.

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The paper is structured as follows. Section 2 provides some mathematical preliminaries, while basic properties of Mazurkiewicz traces are discussed in Section 3. In Section 4 old and new properties of so-structures are discussed. Section 5 is devoted to io-structures and their $\blacklozenge$-closure operator. Section 6 contains some final comments.

2 Relations, Partial Orders and Transitive Closure

In this section, we recall some well-known mathematical concepts and results that will be used frequently in this paper.

Let X be a set and $R_1, R_2 \subseteq X \times X$ are two relations on X. We define $R_1 \circ R_2 \stackrel{df}{=} \{(x,y) \mid \exists z \in Z.\ (x,z) \in R_1 \wedge (y,z) \in R_2\}$, and $id_X \stackrel{df}{=} \{(x,x) \mid x \in X\}$. For each relation $R \subseteq X \times X$, we define R^+, the *transitive closure* of R, as $R^+ \stackrel{df}{=} \bigcup_{i=1}^{\infty} R^i$, and the *reflexive and transitive closure* of R, as $R^* = \bigcup_{i=0}^{\infty} R^i$, where $R^0 = id_X$ and $R^{i+1} = R^i \circ R$ for $i > 0$.

A binary relation $R \subseteq X \times X$ is: *irreflexive* iff for all $a \in X. \neg(a R a)$; *transitive* iff for all $a,b,c \in X.\ a R b \wedge b R c \implies a R c$; and *acyclic* iff for all $a \in X.\ \neg(a R^+ a)$.

A relation $< \subseteq X \times X$ is a (*strict*) *partial order* if it is irreflexive and transitive, i.e. for all $a,c,b \in X$, $a \not< a$ and $a < b < c \implies a < c$. We also define:

$$a \frown_< b \stackrel{df}{\iff} \neg(a < b) \wedge \neg(b < a) \wedge a \neq b$$
$$a <^\frown b \stackrel{df}{\iff} a < b \vee a \frown_< b$$

Note that $a \frown_< b$ means a and b are *incomparable* (w.r.t. $<$) elements of X.

Let $<$ be a partial order on a set x. Then

1. $<$ is *total* if $\frown_< = \emptyset$. In other words, for all $a,b \in X$, $a < b \ \vee\ b < a \ \vee\ a = b$. For clarity, we will reserve the symbol $\lhd$ to denote total orders;
2. $<$ is *stratified* if $a \frown_< b \frown_< c \implies a \frown_< c \vee a = c$, i.e., the relation $\frown_< \cup\, id_X$ is an equivalence relation on X.
3. $<$ is *interval* if for all $a,b,c,d \in X$, $a < c \ \wedge\ b < d \implies a < d \vee b < c$.

It is clear from these definitions that every total order is stratified and every stratified order is interval.

Given a partial order $< \subseteq X \times X$, a relation $<' \subseteq X \times X$ is an *extension* of $<$ if $< \subseteq <'$. For convenience, we define $\mathsf{Total}(<) \stackrel{df}{=} \{\lhd \subseteq X \times X \mid \lhd \text{ is a total order and } < \subseteq \lhd\}$. In other words, the set $\mathsf{Total}(<)$ consists of all the *total order extensions* of $<$.

By Szpilrajn's Theorem [25], we know that every partial order $<$ is uniquely represented by the the set $\mathsf{Total}(<)$. Szpilrajn's Theorem can be stated as following:

Theorem 1 (Szpilrajn [25]). *For every partial order $<$, $< = \bigcap_{\lhd \in \mathsf{Total}(<)} \lhd$.* □

Stratified orders are often defined in an alternative way, namely, a partial order $<$ on X is stratified if and only if there exists a total order $\lhd$ on some Y and a mapping $\phi : X \to Y$ such that $\forall x,y \in X.\ x < y \iff \phi(x) \lhd \phi(y)$. This definition is illustrated in Figure 1, where $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b,c\}$, $\phi(d) = \{d\}$. Note that for all $x,y \in \{a,b,c,d\}$ we have $x <_2 y \iff \phi(x) \lhd_2 \phi(y)$, where the total order $\lhd_2$ can be concisely represented by a *step sequence* $\{a\}\{b,c\}\{d\}$. As a consequence, stratified orders and step sequences can uniquely represent each other (cf. [12, 14, 20]).

For the interval orders, the name and intuition follow from Fishburn's Theorem:

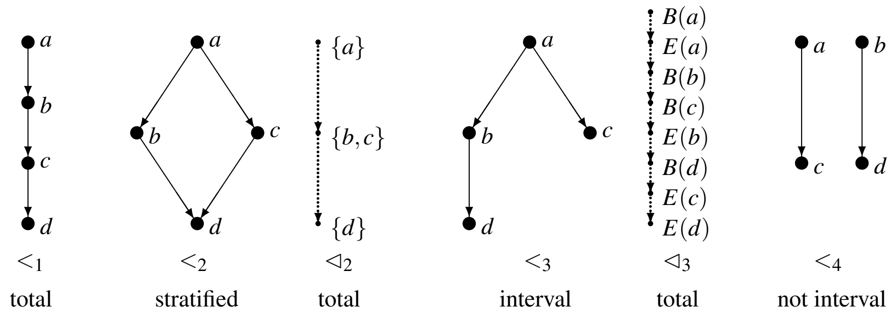


Fig. 1. Various types of partial orders (represented as Hasse diagrams). The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is and extension of $<_4$. Note that order $<_1$, being total, is uniquely represented by a sequence *abcd*, the stratified order $<_2$ is uniquely represented by a step sequence $\{a\}\{b,c\}\{d\}$, and the interval order $<_3$ is (*not* uniquely) represented by a sequence that represents $\lhd_3$, i.e. $B(a)E(a)B(b)B(c)E(b)B(d)E(c)E(d)$.

Theorem 2 (Fishburn [6]). *A partial order $<$ on X is* interval *iff there exists a total order $\lhd$ on some T and two mappings $B, E : X \to T$ such that for all $x, y \in X$,*

1. $B(x) \lhd E(y)$, and 2. $x < y \iff E(x) \lhd B(y)$. □

Usually $B(x)$ is interpreted as the beginning and $E(x)$ as the end of an *interval* x. The intuition of Fishburn's theorem is illustrated in Figure 1 with $<_3$ and $\lhd_3$. For all $x, y \in \{a, b, c, d\}$, we have $B(x) \lhd_3 E(x)$ and $x <_3 y \iff E(x) \lhd_3 B(y)$.

We will next recall the fundamental properties of transitive closure operator.

Proposition 1. *Let $R \subseteq X \times X$.*

1. *If R is irreflexive then $R \subseteq R^+ \setminus id_X$,*
2. $(R^+)^+ = R^+$,
3. *R^+ is a partial order if and only if R^+ is irreflexive,*
4. *if R is a partial order then $R^+ = R$.*
5. *if R is a partial order and $R_0 \subseteq R$, then R_0^+ is a partial order and $R_0^+ \subseteq R$.* □

These properties were extended to the $\diamondsuit$-closure operator for so-structures in [12] and will be extended to the $\blacklozenge$-closure operator for io-structures in Section 5.

3 Partial Orders Generated by Mazurkiewicz Traces

A triple $(X, *, \mathbb{1})$, where X is a set, $*$ is a total binary operation on X, and $\mathbb{1} \in X$, is called a *monoid* [3], if $(a*b)*c = a*(b*c)$ and $a*\mathbb{1} = \mathbb{1}*a = a$, for all $a, b, c \in X$.

A nonempty equivalence relation $\sim \subseteq X \times X$ is a *congruence* in the monoid $(X, *, \mathbb{1})$ if for all $a_1, a_2, b_1, b_2 \in X$, $a_1 \sim b_1 \wedge a_2 \sim b_2 \Rightarrow (a_1 * a_2) \sim (b_1 * b_2)$.

The triple $(X/\!\sim, \circledast, [\mathbb{1}])$, where $[a] \circledast [b] = [a*b]$, is called the *quotient monoid* of $(X, *, 1)$ under the congruence $\sim$. The symbols $*$ and $\circledast$ are often omitted if this does not lead to any discrepancy.

Let $M = (X, *, \mathbb{1})$ be a *monoid* and let $EQ = \{\ x_i = y_i \mid i = 1, \dots, n\ \}$ be a finite set of *equations*. Define $\equiv_{EQ}$ (or just $\equiv$) to be the *least congruence* on M satisfying, $x_i = y_i \implies x_i \equiv_{EQ} y_i$, for each equation $x_i = y_i \in EQ$. We call the relation $\equiv_{EQ}$ the *congruence defined by EQ*, or *EQ-congruence*.

The *quotient monoid* $M_{\equiv_{EQ}} = (X/\equiv_{EQ}, \circledast, [\mathbb{1}])$, where $[x] \circledast [y] = [x * y]$, is called an *equational monoid* (see [14, 20] for more details).

Monoids of *Mazurkiewicz traces* (or *traces*) (cf. [5, 21]) are *equational monoids over sequences*. The theory of traces has been utilised to tackle problems from quite diverse areas including combinatorics, graph theory, algebra, logic and, especially concurrency theory [5, 21].

Applications of traces in concurrency theory are originated from the fact that traces are *sequence representation of partial orders*, which gives traces the ability to model "true concurrency" semantics. We will now recall the definition of a *trace monoid*.

Definition 1 ([5, 21]). *Let $M = (E^*, *, \lambda)$ be a* free monoid *generated by E, and let the relation $ind \subseteq E \times E$ be an irreflexive and symmetric relation (called* independency*), and $EQ = \{ab = ba \mid (a,b) \in ind\}$. Let $\equiv_{ind}$, called* trace congruence*, be the congruence defined by EQ. Then the equational monoid $M_{\equiv_{ind}} = \left(E^*/\equiv_{ind}, \circledast, [\lambda]\right)$ is a monoid of* traces*. The pair (E, ind) is called a* trace alphabet*.*

We will omit the subscript *ind* from trace congruence if it causes no ambiguity.

Example 1. Let $E = \{a, b, c\}$, $ind = \{(b, c), (c, b)\}$, i.e., $EQ = \{\ bc = cb\ \}$. Given three sequences $s = abcbca$, $s_1 = abc$ and $s_2 = bca$, we can generate the traces $[s] = \{abcbca, abccba, acbbca, acbcba, abbcca, accbba\}$, $[s_1] = \{abc, acb\}$ and $[s_2] = \{bca, cba\}$. Note that $[s] = [s_1] \circledast [s_2]$ since $[abcbca] = [abc] \circledast [bca] = [abc * bca]$.

Each trace represents a finite partial order in the following sense. For the trace $[s]$ from Example 1, we can define $\Sigma_{[s]} = \left\{a^{(1)}, b^{(1)}, c^{(1)}, b^{(2)}, c^{(2)}, a^{(2)}\right\}$ to be the set of all *enumerated events* occurring in $[s]$, where $a^{(1)}$ and $a^{(2)}$ simply denote the first and the second occurrences of a respectively in the sequence s_1. Then the partially ordered set (poset) $\left(\Sigma_{[s]}, \prec_{[s]}\right)$ represented by $[s]$ is depicted in the diagram on the right (arcs inferred from transitivity are omitted for simplicity).

$$\begin{tikzcd}
 & b^{(1)} \arrow[r] & b^{(2)} \arrow[dr] & \\
a^{(1)} \arrow[ur] \arrow[dr] & & & a^{(2)} \\
 & c^{(1)} \arrow[r] & c^{(2)} \arrow[ur] &
\end{tikzcd}$$

In fact, the total orders induced by the elements of $[s]$ comprise *all* the total extensions of $\prec_{[s]}$ (see [21]), which by Theorem 1 implies that $[s]$ *uniquely determines* the partial order $\prec_{[s]}$.

Remark 1. Given a sequence s, to construct the partial order $\prec_{[s]}$ represented by $[s]$, we *do not* need to build up to exponentially many elements of $[s]$. We can simply construct the direct acyclic graph $(\Sigma_{[s]}, \prec_s)$, where $x^{(i)} \prec_s y^{(j)}$ iff $x^{(i)}$ occurs before $y^{(j)}$ on the sequence s and $(x, y) \notin ind$. The relation $\prec_s$ is usually *not* the same as the partial order $\prec_{[s]}$. However, after applying the *transitive closure* operator, we have $\prec_{[s]} = \prec_s^+$. To extend this simple idea to the more difficult cases of constructing stratified or io-structures from their sequence representations, it is inevitable that we have to generalise the *transitive closure* operator to these order structures.

4 Stratified Order Structures, Comtraces, and $\diamondsuit$-Closure

A *relational structure* is a triple $S = (X, R_1, R_2)$, where $R_1, R_2 \subseteq X \times X$. We will write $S = (X, R_1, R_2) \subseteq S' = (X, R'_1, R'_2)$ iff $R_1 \subseteq R'_1$ and $R_2 \subseteq R'_2$.

Definition 2 ([11]). *A* stratified order structures *(*so-structure*) is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c \in X$, the following hold:*

S1:	$a \not\sqsubset a$	*S3:*	$a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$
S2:	$a \prec b \implies a \sqsubset b$	*S4:*	$a \sqsubset b \prec c \vee a \prec b \sqsubset c \implies a \prec c$

So-structures were independently introduced in [8] and [10]. Their comprehensive theory has been presented in [12, 13]. They have been successfully applied to model inhibitor and priority systems, asynchronous races, synthesis problems, etc., [17] (see [9] for more references).

The relation $\prec$ is called *causality* and represents the "earlier than" relationship, and the relation $\sqsubset$ is called *weak causality* and represents the "not later than" relationship. The axioms S1–S4 model the mutual relationship between "earlier than" and "not later than" relations, provided that the system runs are defined as stratified orders.

A stratified order $<$ on X is a *stratified extension* of a so-structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^\frown$. The set of all stratified extensions of S will be denoted by $\mathsf{Strat}(S)$.

Theorem 3 ([13]). *For every so-structure $S = (X, \prec, \sqsubset)$:*

$$S = \Big(X, \bigcap_{<\in \mathsf{Strat}(S)} <, \bigcap_{<\in \mathsf{Strat}(S)} <^\frown\Big). \qquad \square$$

The above theorem is a generalisation of Szpilrajn's Theorem to so-structures and is interpreted as the proof of the claim that so-structures uniquely represent sets of equivalent system runs provided that the system operational semantics can be fully described in terms of stratified orders (see [9, 13] for details).

We will now present the concept of $\diamondsuit$-closure that plays a substantial role in most of the applications of so-structures for modelling concurrent systems (cf. [13, 16, 17]).

Definition 3 ([12]). *For every relational structure $S = (X, R_1, R_2)$ we define $S^\diamondsuit$ as*

$$S^\diamondsuit \stackrel{df}{=} \Big(X, \prec^\diamondsuit_{R_1,R_2}, \sqsubset^\diamondsuit_{R_1,R_2}\Big) = \big(X, (R_1 \cup R_2)^* \circ R_1 \circ (R_1 \cup R_2)^*, (R_1 \cup R_2)^* \setminus id_X\big).$$

Intuitively the $\diamondsuit$-closure is a generalisation of transitive closure for relations to so-structures. The theorem below shows that the $\diamondsuit$-closure has all the properties formulated for transitive closure in Proposition 1.

Theorem 4 ([12]). *Let $S = (X, R_1, R_2)$ be a relational structure.*

1. *If R_2 is irreflexive then $S \subseteq S^\diamondsuit$.*
2. *$(S^\diamondsuit)^\diamondsuit = S^\diamondsuit$.*
3. *$S^\diamondsuit$ is a so-structure if and only if the relation $\prec^\diamondsuit_{R_1,R_2}$ is irreflexive.*
4. *If S is a so-structure then $S = S^\diamondsuit$.*
5. *Let S be a so-structure and let $S_0 \subseteq S$. Then $S_0^\diamondsuit \subseteq S$ and $S_0^\diamondsuit$ is a so-structure.* $\square$

Among others, Theorem 4 helps us to show a relationship between so-structures and *comtraces*, an extension of Mazurkiewicz traces that allows us to model the "not later than" relationship using quotient monoids of *step sequence* monoids [12, 14, 20].

Definition 4 ([12]). *Let E be a finite set (of events) and let $ser \subseteq sim \subset E \times E$ be two relations called* serialisability *and* simultaneity *respectively and the relation sim is irreflexive and symmetric. Then the triple (E, sim, ser) is called the* comtrace alphabet.

Intuitively, if $(a,b) \in sim$ then a and b can occur simultaneously, while $(a,b) \in ser$ means that a and b may occur simultaneously or a may occur before b (i.e., both executions are equivalent). We define $\mathbb{S}$, the set of all (potential) *steps*, as the set of all cliques of the graph (E, sim), i.e., $\mathbb{S} \overset{df}{=} \{A \mid A \neq \emptyset \wedge \forall a,b \in A.\ (a = b \vee (a,b) \in sim)\}$. Hence, the triple $(\mathbb{S}^*, *, \lambda)$, where "$*$" denotes the step sequence concatenation operator (usually omitted), is a *monoid of step sequences*.

Definition 5 ([12]). *Let $\theta = (E, sim, ser)$ be a comtrace alphabet and let $\equiv_{ser}$, called* comtrace congruence*, be the EQ-congruence defined by the set of equations:*

$$EQ = \{A = BC \mid A = B \cup C \in \mathbb{S} \wedge B \times C \subseteq ser\}.$$

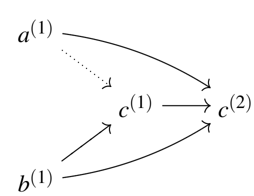
Then the equational monoid $(\mathbb{S}^*/\equiv_{ser}, \circledast, [\lambda])$ is called a monoid of comtraces *over θ.*

We will omit the subscript *ser* from comtrace congruence if it causes no ambiguity.

Example 2. Let $E = \{a,b,c\}$, $sim = \{(a,b),(b,a),(a,c),(c,a)\}$ and $ser = \{(a,b),(b,a),(a,c)\}$. Then we have $\mathbb{S} = \{\{a\},\{b\},\{c\},\{b,c\}\}$. A step sequence $s = \{a,b\}\{c\}\{c\}$ generates $[s] = \{\{a,b\}\{c\}\{c\}, \{a\}\{b\}\{c\}\{c\}, \{b\}\{a\}\{c\}\{c\}, \{b\}\{a,c\}\{c\}\}$ as its comtrace. Note that $\{a\}\{c\}\{b\}\{a\} \notin [s]$.

Let $u = A_1 \ldots A_k$ be a step sequence. By $\overline{u} = \overline{A}_1 \ldots \overline{A}_k$ be the *event enumerated* representation of u. We will skip a lengthy but intuitively obvious formal definition (cf. [12]), but for instance, from Example 2, $\overline{s} = \{a^{(1)}, b^{(1)}\}\{c^{(1)}\}\{c^{(2)}\}$. Let $\Sigma_u = \bigcup_{i=1}^{k} \overline{A}_i$ denote the set of all enumerated events occurring in u, for example, $\Sigma_s = \{a^{(1)}, b^{(1)}, c^{(1)}, c^{(2)}\}$. For each $\alpha \in \Sigma_u$, let $pos_u(\alpha)$ denote the consecutive number of a step where α belongs, i.e. if $\alpha \in \overline{A}_j$ then $pos_u(\alpha) = j$. For our example, $pos_s(c^{(2)}) = 3$, $pos_s(b^{(1)}) = 1$, etc. For each enumerated even $\alpha = e^{(i)}$, let $l(\alpha)$ denote the *label* of α, i.e. $l(\alpha) = l(e^{(i)}) = e$. One can easily show that $u \equiv v \implies \Sigma_u = \Sigma_v$, so we can define $\Sigma_{[u]} = \Sigma_u$.

Given a step sequence u, we define the stratified order $\lhd_u \subseteq \Sigma_u \times \Sigma_u$ induced by u by: $\alpha \lhd_u \beta \overset{df}{\iff} pos_u(\alpha) < pos_u(\beta)$. Then it can be easily checked that the stratified orders induced by the step sequences of the comtrace $[s]$ from Example 2 are exactly the stratified extensions of the so-structure $S_{[s]} = (\Sigma_{[s]}, \prec_{[s]}, \sqsubset_{[s]})$ on the right. The dotted edge denotes $\sqsubset_{[s]}$, while the solid edges denote both $\prec_{[s]}$ and $\sqsubset_{[s]}$.



Analogous to Remark 1 for traces, given a comtrace alphabet (E, sim, ser) and a step sequence u, we do *not* need to analyse any other elements of $[u]$ except u itself to construct the so-structure $S_{[u]}$, which the comtrace $[u]$ represents. We will now show how the $\Diamond$-closure operator helps us to build the desired construction.

Definition 6 **([12]).** *Let $u \in \mathbb{S}^*$. We define the relations $\prec_u, \sqsubset_u \subseteq \Sigma_{[u]} \times \Sigma_{[u]}$ as:*

1. $\alpha \prec_u \beta \stackrel{df}{\iff} \alpha \lhd_u \beta \wedge (l(\alpha), l(\beta)) \notin ser$,
2. $\alpha \sqsubset_u \beta \stackrel{df}{\iff} \alpha \lhd_u^{\frown} \beta \wedge (l(\beta), l(\alpha)) \notin ser$.

Definition 6 describes two basic *"local"* invariants of the elements of $\Sigma_{\mathbf{u}}$. The relation $\prec_u$ captures the situation when α *always precedes* β, and the relation $\sqsubset_u$ captures the situation when α *never follows* β. However, since $\prec_u$ and $\sqsubset_u$ are "locally" invariant, the relation structure $(\Sigma_{[u]}, \prec_u, \sqsubset_u)$ might not contain "global" invariants that can be inferred from (S3) and (S4) of Definition 2. For instance, the step sequence s from Example 2 generates the following relations $\prec_s = \{(b^{(1)}, c^{(1)}), (b^{(1)}, c^{(2)}), (c^{(1)}, c^{(2)})\}$ and $\sqsubset_s = \prec_s \cup \{(a^{(1)}, c^{(1)}), (a^{(1)}, c^{(2)})\}$, where the edge $(a^{(1)}, c^{(2)})$ from $\prec_{[s]}$ is absent from $\prec_s$. To make sure all invariants are included, we need $\Diamond$-closure.

Definition 7. *Given a step sequence $u \in \mathbb{S}^*$ and its respective comtrace $[u] \in \mathbb{S}^*/\equiv$. We define the relational structures $S_{[u]}$ as:* $S_{[u]} \stackrel{df}{=} \big(\Sigma_{[u]}, \prec_u, \sqsubset_u\big)^{\Diamond}$.

The relational structure $S_{[u]}$ is the so-structure defined by the comtrace $[u]$. The following theorem justifies the names and summarises the following nontrivial results concerning the so-structures generated by comtraces. The proofs of these results heavily use the properties of $\Diamond$-closure from Theorem 4.

Theorem 5 **([12, 13]).** *For all $u, v \in \mathbb{S}^*$, we have*

1. *$S_{[u]}$ is a so-structure and* $S_{[u]} = \Big(\Sigma_{[u]}, \bigcap_{x\in[u]} \lhd_x, \bigcap_{x\in[u]} \lhd_x^{\frown}\Big)$,
2. $u \equiv v \iff S_{[u]} = S_{[v]}$,
3. $ext\big(S_{[u]}\big) = \big\{\lhd_s \mid s \in [u]\big\}$. □

Note that a generalisation of Theorem 5 to *generalised stratified order structures* (*gso-structures*) [9], an extension of so-structures which can additionally model the "non-simultaneously" relationship, was recently shown in [14, 20]. A sequence representation of gso-structures called *generalised comtraces* were proposed and shown to represent precisely finite gso-structures. The intuition of the approach in [20] is similar to what we discussed here and the $\Diamond$-closure operator was applied extensively.

5 Interval Order Structures and ⧫-closure

This section contains the major contribution of this paper. We start with a short presentation of some properties on io-structures, then we define ⧫-closure, the main concept of this paper, and prove the equivalence of Theorem 4. Because io-structures are more complex than so-structures, the proofs are more involved than that of Theorem 4.

Definition 8 **([11]).** *An* interval order structure *(io-structure) is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c, d \in X$, the following hold:*

I1: $a \not\sqsubset a$	*I4:* $a \prec b \sqsubset c \vee a \sqsubset b \prec c \implies a \sqsubset c$
I2: $a \prec b \implies a \sqsubset b$	*I5:* $a \prec b \sqsubset c \prec d \implies a \prec d$
I3: $a \prec b \prec c \implies a \prec c$	*I6:* $a \sqsubset b \prec c \sqsubset d \implies a \sqsubset d \vee a = d$

Here the *causality* relation $\prec$ also represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than" relationship but under the assumption that the system runs are interval orders.

Proposition 2 ([11]).

1. *$\prec$ is a partial order such that $a \prec b \Rightarrow b \not\sqsubset a$ and $a \sqsubset b \Rightarrow b \not\prec a$.*
2. *If $<$ is an interval order on X, then $(X, <, <^\frown)$ is an io-structure.* $\square$

Interval order structures were independently introduced in [19] and [10]. Some of their properties have been presented in [13], yet their theory is not as well-developed and much less often applied than that of so-structures [9]. The lack of an operator analogous to the $\Diamond$-closure prevented us from building a working relationship between io-structures and sequence models of concurrency such as Mazurkiewicz traces and comtraces.

Theorem 6 ([13]). *Every so-structure is an io-structure.* $\square$

Since every so-structure is an io-structure, many properties of so-structures also hold for io-structures. Furthermore, we also have an analogue of Theorem 3 for interval orders and io-structures.

An interval order $<$ on X is an *interval extension* of an io-structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^\frown$. The set of all interval extensions of S will be denoted by $\mathsf{Interv}(S)$.

Theorem 7 ([13]). *For each io-structure $S = (X, \prec, \sqsubset)$, we have*

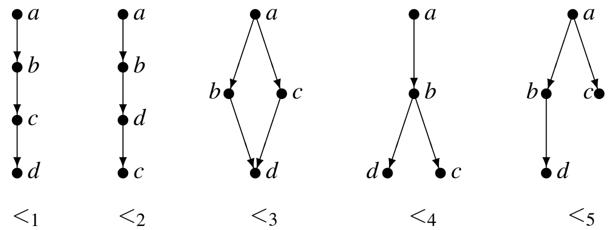
$$S = \Big(X, \bigcap_{<\in \mathsf{Interv}(S)} <, \bigcap_{<\in \mathsf{Interv}(S)} <^\frown\Big).$$ $\square$

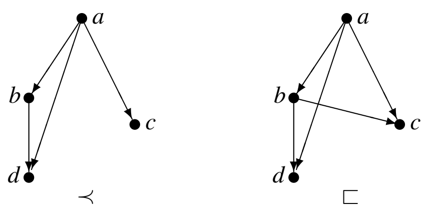
The above theorem is a generalisation of Szpilrajn's Theorem to io-structures. It is interpreted as the proof of the claim that io-structures uniquely represent sets of equivalent system runs, provided that the system's operational semantics can be fully described in terms of interval orders (see [9, 13] for details). An example of a simple interval order structure which illustrates the main ideas behind this concept is shown in Figure 2.

Before defining the concept of $\blacklozenge$-closure and proving its properties, we need to introduce some auxiliary notions and prove some preliminary results.

Definition 9. *Let $R_1, R_2 \subseteq X \times X$ be two relations and let $\langle S_1, \ldots, S_k \rangle$ be a sequence of relations such that $S_i \in \{R_1, R_2\}$, $i = 1, \ldots, k$.*

1. *A sequence $\langle S_1, \ldots, S_k \rangle$ has $\uplus$-property w.r.t. (R_1, R_2), if for all i, $1 \le i < k$, we have $\neg(S_i = S_{i+1} = R_2)$, i.e. there are no two consecutive R_2's.*
2. *A sequence $\langle S_1, \ldots, S_k \rangle$ has $\oplus$-property w.r.t. (R_1, R_2), if $k \ge 1$, $S_1 = S_k = R_1$ and the sequence $\langle S_2, \ldots, S_{k-1} \rangle$ has $\uplus$-property w.r.t. (R_1, R_2);*
3. *$R_1 \uplus R_2 = \bigcup_{k \ge 0} \{S_1 \circ \ldots \circ S_k \mid \langle S_1, \ldots, S_k \rangle$ has $\uplus$-property w.r.t. $(R_1, R_2)\}$.*
4. *$R_1 \oplus R_2 = \bigcup_{k \ge 1} \{S_1 \circ \ldots \circ S_k \mid \langle S_1, \ldots, S_k \rangle$ has $\oplus$-property w.r.t. $(R_1, R_2)\}$.*






```
P: begin int x,y,z:
      a: begin x:=0; y:=0; z:=0 end;
      cobegin
        begin
          b: x=0 → y:=y+1;
          d: z:=z+1
        end,
          c: x:=x+1
      coend
    end P
```

Fig. 2. An example of a simple interval order structure $S = (X, \prec, \sqsubset)$, with $X = \{a,b,c,d\}$ and its set of all interval extensions $\mathsf{Interv}(S) = \{<_1, <_2, <_3, <_4, <_5\}$. The orders $<_1$ and $<_2$ are total, $<_3$ and $<_4$ are stratified and $<_5$ is interval but not stratified. The elements of $\mathsf{Interv}(S)$ are all equivalent runs (executions) of the program P involving the actions a, b, c and d, so the interval order structure uniquely defines a concurrent behaviour (history) of P (see [9] for details). The elements of $\mathsf{Interv}(S)$ are represented as Hasse diagrams, while $\prec$ and $\sqsubset$ are represented as graphs of their entire relations. In this case $\prec$ equals $<_5$, as there are not so many partial orders over the four elements set, but the interpretations of $<_5$ and $\prec$ are different. The incomparability in $<_5$ is interpreted as *simultaneity* while in $\prec$ as *having no casual relationship*.

For example the sequence $\langle R_1, R_2, R_2, R_1\rangle$ has neither $\uplus$- nor $\oplus$-property, the empty sequence $\langle\rangle$ and the sequence $\langle R_1, R_1, R_2, R_1, R_2\rangle$ has $\uplus$-property but not $\oplus$-property, and $\langle R_1, R_1, R_2, R_1, R_2, R_1\rangle$ has $\oplus$-property. We will omit the suffix "w.r.t. (R_1, R_2)" if the relations R_1 and R_2 are clear from the context. The relations $R_1 \oplus R_2$ and $R_1 \uplus R_2$ can easily be defined by appropriate regular expressions built from R_1 and R_2.

Proposition 3. *Let $R_1, R_2 \subseteq X \times X$ be two relations. Then*
1. $R_1 \oplus R_2 = (R_1^+ \circ R_2)^* \circ R_1^+$,
2. $R_1 \uplus R_2 = (R_2 \cup id_X) \circ (R_1 \cup R_1 \circ R_2)^*$,
3. $R_1 \oplus R_2 \subseteq R_1 \uplus R_2$,
4. $(R_1 \oplus R_2) \uplus (R_1 \uplus R_2) \subseteq (R_1 \uplus R_2)$,
5. $(R_1 \oplus R_2) \oplus (R_1 \uplus R_2) \subseteq (R_1 \oplus R_2)$.

Proof. Follows immediately from Definition 9. □

We can now define the main concept of this paper, the concept of ♦*-closure*.

Definition 10. *For every relational structure $S = (X, R_1, R_2)$ we define $S^{\blacklozenge}$, the* ♦-closure *of S, as:*

$$S^{\blacklozenge} \stackrel{df}{=} (X, \prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2}) = (X, R_1 \oplus R_2, (R_1 \uplus R_2) \setminus id_X).$$

The $\blacklozenge$-closure is an extension of $\lozenge$-closure of so-structures and transitive closure of relations to io-structures. We will start by proving equivalences of Theorem 4(1,2).

Proposition 4. *1. If R_2 is irreflexive, then $S \subseteq S^{\blacklozenge}$.* *2. $\left(S^{\blacklozenge}\right)^{\blacklozenge} = S^{\blacklozenge}$.*

Proof. 1. By the definition $R_1 \subseteq R_1 \oplus R_2 = \prec^{\blacklozenge}_{R_1,R_2}$ and $R_2 \subseteq R_1 \uplus R_2$. Hence, if R_2 is irreflexive, $R_2 \setminus id_X \subseteq (R_1 \uplus R_2) \setminus id_X = \sqsubset^{\blacklozenge}_{R_1,R_2}$.

2. ($\supseteq$) Since $\sqsubset_{R_1,R_2}$ is irreflexive, by (1) we have $S^{\blacklozenge} \subseteq (S^{\blacklozenge})^{\blacklozenge}$.
($\subseteq$) We need to show that $\prec^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} \subseteq \prec^{\blacklozenge}_{R_1,R_2}$ and $\sqsubset^{\blacklozenge}_{\prec^{\blacklozenge}_{R_1,R_2},\sqsubset^{\blacklozenge}_{R_1,R_2}} \subseteq \sqsubset^{\blacklozenge}_{R_1,R_2}$, which means $(R_1 \oplus R_2) \oplus (R_1 \uplus R_2) \subseteq R_1 \oplus R_2$, and $(R_1 \oplus R_2) \uplus (R_1 \uplus R_2) \subseteq R_1 \uplus R_2$. But this follows from Proposition 3(4,5). □

Proposition 4(2) states that $\blacklozenge$-closure is *idempotent*, and justifies the name *closure* (cf. [24]).

Note that the exact replica of Theorem 4(3) is false. Consider an example, where $X = \{a,b\}$, $R_1 = \{(a,b)\}$ and $R_2 = \{(b,a)\}$. Thus, $\prec^{\blacklozenge}_{R_1,R_2} = \{(a,b)\}$ and $\sqsubset^{\blacklozenge}_{R_1,R_2} = \{(a,b),(b,a)\}$, so $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive, but $(X, \prec^{\blacklozenge}_{R_1,R_2}, \sqsubset^{\blacklozenge}_{R_1,R_2})$ is not an io-structure since $a \prec^{\blacklozenge}_{R_1,R_2} b \sqsubset^{\blacklozenge}_{R_1,R_2} a \implies a \sqsubset a$, which contradicts (I1) from Definition 8. To find the necessary and sufficient condition for the $\blacklozenge$-closure of a relational structure to be an io-structure, we need a new concept.

Definition 11. *A relational structure $S = (X, R_1, R_2)$ is* i-directed *if*

1. *$R_1 \oplus R_2$ is irreflexive, and*
2. *$\forall a,b \in X.\ (a,b) \in R_2 \implies (b,a) \notin R_1 \oplus R_2$.*

Proposition 5. *$S^{\blacklozenge}$ is an io-structure if and only if $S = (X, R_1, R_2)$ is i-directed.*

Proof. ($\Rightarrow$) If $S^{\blacklozenge}$ is an io-structure then by (I1) and (I2), $\prec^{\blacklozenge}_{R_1,R_2} = R_1 \oplus R_2$ is irreflexive. Suppose $(a,b) \in R_2$ and $(b,a) \in R_1 \oplus R_2$. Since $R_2 \subseteq \sqsubset^{\blacklozenge}_{R_1,R_2}$, we have $a \prec^{\blacklozenge}_{R_1,R_2} b$ and $b \sqsubset^{\blacklozenge}_{R_1,R_2} a$, which contradicts Proposition 2(1).
($\Leftarrow$) We need to show that the conditions of Definition 8 are satisfied.

(I1) Clearly $(R_1 \uplus R_2) \setminus id_X$ is irreflexive.

(I2) From Corollary 3(3) we have $\prec^{\blacklozenge}_{R_1,R_2} \subseteq R_1 \uplus R_2$. Since $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive, $\prec^{\blacklozenge}_{R_1,R_2} \subseteq (R_1 \uplus R_2) \setminus id_X = \sqsubset^{\blacklozenge}_{R_1,R_2}$.

(I3) Let $a \prec^{\blacklozenge}_{R_1,R_2} b$ and let $b \prec^{\blacklozenge}_{R_1,R_2} c$. This means $aS_1 \circ \ldots \circ S_k b Q_1 \circ \ldots \circ Q_r c$, where $\langle S_1, \ldots, S_k \rangle$ and $\langle Q_1, \ldots, Q_r \rangle$ both have $\oplus$-property. Hence $\langle S_1, \ldots, S_k, Q_1, \ldots, Q_r \rangle$ also has $\oplus$-property. Thus, $a \prec_{R_1,R_2} c$.

(I4) Let $a \prec^{\blacklozenge}_{R_1,R_2} b$ and let $b \sqsubset^{\blacklozenge}_{R_1,R_2} c$. This means $aS_1 \circ \ldots \circ S_k b Q_1 \circ \ldots \circ Q_r c$, where $\langle S_1, \ldots, S_k \rangle$ satisfies $\oplus$-property, and $\langle Q_1, \ldots, Q_r \rangle$ satisfies $\uplus$-property. Hence the sequence $\langle S_1, \ldots, S_k, Q_1, \ldots, Q_r \rangle$ has $\uplus$-property and thus $(a,c) \in R_1 \uplus R_2$. Suppose $a = c$. Since $a \prec^{\blacklozenge}_{R_1,R_2} b$ and $b \sqsubset^{\blacklozenge}_{R_1,R_2} c$, this means $aR_1 \circ S_1 \circ \ldots \circ S_k \circ R_1 b$, and $bQ_1 \circ Q_2 \circ \ldots \circ Q_{s-1} \circ Q_s a$, where $S_i, Q_i \in \{R_1, R_2\}$. Either Q_1 or Q_s are equal to R_2, otherwise $b \prec^{\blacklozenge}_{R_1,R_2} a$, contradicting that $\prec^{\blacklozenge}_{R_1,R_2}$ is irreflexive. Suppose $Q_1 = R_2$. This means $Q_2 = R_1$. Thus there is some b_1 such that $bR_2b_1R_1 \circ Q_3 \circ \ldots \circ Q_s \circ R_1 \circ S_1 \circ \ldots \circ S_k \circ R_1 b$, which means $(b_1, b) \in R_1 \oplus R_2$, contradicting Definition 11(2). Hence $Q_1 = R_1$ and $Q_s = R_2$, i.e. $Q_{s-1} = R_1$. Thus, there is some b_s such that b_sR_2a and $aR_1 \circ S_1 \circ \ldots \circ S_k \circ R_1 \circ Q_1 \circ \ldots \circ R_1 b_s$, which means $(a, b_s) \in R_1 \oplus R_2$, contradicting Definition 11(2). Therefore $a \neq c$, i.e. $(a,c) \in (R_1 \uplus R_2) \setminus id_X = \sqsubset^{\blacklozenge}_{R_1,R_2}$.

For the case when $a \sqsubset^{\blacklozenge}_{R_1,R_2} b \prec^{\blacklozenge}_{R_1,R_2} c$, we proceed almost identically.

(I5) Let $a \prec^{\blacklozenge}_{R_1,R_2} b \sqsubset^{\blacklozenge}_{R_1,R_2} c \prec^{\blacklozenge}_{R_1,R_2} d$. Thus, there are sequences $\langle S_1, \ldots, S_k\rangle$, $\langle P_1, \ldots, P_s\rangle$ and $\langle Q_1, \ldots, Q_r\rangle$, such that $aS_1 \circ \ldots \circ S_k bP_1 \circ \ldots \circ P_s cQ_1 \circ \ldots \circ Q_r d$, where $\langle S_1, \ldots, S_k\rangle$ and $\langle Q_1, \ldots, Q_r\rangle$ have $\oplus$-property and $\langle P_1, \ldots, P_s\rangle$ has $\uplus$-property. It follows that $\langle S_1, \ldots, S_k, Q_1, \ldots, Q_r, P_1, \ldots, P_s\rangle$ has $\oplus$-property and thus $a \prec^{\blacklozenge}_{R_1,R_2} d$.

(I6) Let $a \sqsubset^{\blacklozenge}_{R_1,R_2} b \prec^{\blacklozenge}_{R_1,R_2} c \sqsubset^{\blacklozenge}_{R_1,R_2} d$. Thus, there are sequences $\langle S_1, \ldots, S_k\rangle$, $\langle P_1, \ldots, P_s\rangle$ and $\langle Q_1, \ldots, Q_r\rangle$, such that $aS_1 \circ \ldots \circ S_k bP_1 \circ \ldots \circ P_s cQ_1 \circ \ldots \circ Q_r d$, where $\langle S_1, \ldots, S_k\rangle$ and $\langle Q_1, \ldots, Q_r\rangle$ have $\uplus$-property and $\langle P_1, \ldots, P_s\rangle$ has $\oplus$-property. It follows that $\langle S_1, \ldots, S_k, Q_1, \ldots, Q_r, P_1, \ldots, P_s\rangle$ has $\uplus$-property. So $a \sqsubset^{\blacklozenge}_{R_1,R_2} b$ or $a = d$. $\square$

The fact that the above result is slightly weaker than Theorem 4(3) does not seem to matter much as in virtually all applications of $\lozenge$-closure in [12] and [17], the relations R_1 and R_2 satisfy the equivalence of the conditions of Definition 11 for so-structures. The below result appears to be quite useful for various potential applications of $\blacklozenge$-closure.

Proposition 6. *Let $S = (X, R_1, R_2)$ be a relational structure and let $<\subseteq X \times X$ be an interval order such that $R_1 \subseteq <$ and $R_2 \subseteq <^{\frown}$. Then S is i-directed.*

Proof. By Proposition 2(2), $(X, <, <^{\frown})$ is an io-structure, so it satisfies I1–I6. We have $R_1^+ \subseteq <^+ = <$, so $R_1 \oplus R_2 = (R_1^+ \circ R_2)^* \circ R_1^+ \subseteq (< \circ R_2)^* \circ < = \bigcup_{i=0}^{\infty}((< \circ R_2)^i \circ <)$. For each i, we have $(< \circ R_2)^i \circ < \subseteq (< \circ <^{\frown})^i \circ <$ and then by applying (I5) i times, we have $(< \circ <^{\frown})^i \circ < \subseteq <$. Hence $R_1 \oplus R_2 \subseteq <$. i.e. $R_1 \oplus R_2$ is irreflexive. If $(a,b) \in R_2$ then $a <^{\frown} b$, i.e. $\neg(b < a)$ and also $(a,b) \notin R_1 \oplus R_2$ as $R_1 \oplus R_2 \subseteq <$. $\square$

Both $\lozenge$- and $\blacklozenge$-closures are often used for the cases like the one in Definition 7, so we can then use the above results to simplifies the proofs.

We now prove an analogue of Theorem 4(4), which states that io-structures are fixed points of $\blacklozenge$-closure.

Proposition 7. *If $S = (X, \prec, \sqsubset)$ is an io-structure then $S = S^{\blacklozenge}$.*

Proof. ($\subseteq$) Since S is an io-structure, $\sqsubset$ is irreflexive. Thus, by Proposition 4(1), $S \subseteq S^{\blacklozenge}$.
($\supseteq$) We will first show that $\prec \oplus \sqsubset \subseteq \prec$. Since $\prec \oplus \sqsubset = (\prec^+ \circ \sqsubset)^* \circ \prec^+$, it suffices to show that for each $i \geq 1$, $j \geq 0$, $k \geq 1$, $(\prec^i \circ \sqsubset)^j \circ \prec^k \subseteq \prec$. From (I3) it follows $\prec^i \subseteq \prec$ and $\prec^k \subseteq \prec$, so $(\prec^i \circ \sqsubset)^j \circ \prec^k \subseteq (\prec \circ \sqsubset)^j \circ \prec$. By apply (I5) from right to left i times, we have $(\prec \circ \sqsubset)^j \circ \prec \subseteq \prec$. Thus, $\prec \oplus \sqsubset \subseteq \prec$.

It remains to show $(\prec \uplus \sqsubset) \setminus id_X \subseteq \sqsubset$. By Proposition 3(2), $\prec \uplus \sqsubset = (\sqsubset \cup id_X) \circ (\prec \cup \prec \circ \sqsubset)^*$. It suffices to show that for all $i \geq 0$, $(\sqsubset \cup id_X) \circ (\prec \cup \prec \circ \sqsubset)^i \subseteq \sqsubset \cup id_X$. The case when $i = 0$ is trivial. For $i > 0$, by the induction hypothesis, we have $(\sqsubset \cup id_X) \circ (\prec \cup \prec \circ \sqsubset)^{i-1} \subseteq \sqsubset \cup id_X$. It suffices to show $(\sqsubset \cup id_X) \circ (\prec \cup \prec \circ \sqsubset) \subseteq \sqsubset \cup id_X$. But this holds since, by (I4) and (I6), $\big((\sqsubset \cup id_X) \circ \prec\big) \cup \big((\sqsubset \cup id_X) \circ \prec \circ \sqsubset\big) \subseteq \sqsubset \cup id_X$. $\square$

Directly from Proposition 7 we obtain the below result which will be used in the proof of the analogue of Theorem 4(5).

Corollary 1. *Every io-structure is i-directed.* □

Proposition 8. *Let $S = (X, \prec, \sqsubset)$ be an io-structure and let $S_0 \subseteq S$. Then $S_0^{\blacklozenge} \subseteq S$ and $S_0^{\blacklozenge}$ is an io-structure.*

Proof. From Proposition 7 it immediately follows $S_0^{\blacklozenge} \subseteq S^{\blacklozenge} = S$.

Due to Proposition 5 it suffices to show that S_0 is i-directed. Let $S_0 = (X, R_1, R_2)$. We have $R_1 \oplus R_2 \subseteq \prec \oplus \sqsubset =^{\text{(Proposition 7)}} \prec$. Since $\prec$ is irreflexive, $R_1 \oplus R_2$ is irreflexive as well. Let $(a,b) \in R_2$. Since $R_2 \subseteq \sqsubset$, we have $a \sqsubset b$ which by Corollary 1, implies $(b,a) \notin \prec \oplus \sqsubset$. Since $R_1 \oplus R_2 \subseteq \prec \oplus \sqsubset$, $(b,a) \notin R_1 \oplus R_2$. Therefore S_0 is i-directed. □

We can also show that $\blacklozenge$-closure is indeed a generalisation of $\lozenge$-closure.

Proposition 9. *If S is so-structure then $S = S^{\lozenge} = S^{\blacklozenge}$.*

Proof. A consequence of Theorem 4(4), Theorem 6 and Proposition 7. □

6 Final Comments

A concept of $\blacklozenge$-closure has been defined for io-structures. It is an equivalence of $\lozenge$-closure of so-structures ([12]) and classical transitive closure of relations. It has also been proven that, in principle, $\blacklozenge$-closure has the same properties as $\lozenge$-closure and transitive closure. Because the definition of $\blacklozenge$-closure was more elaborate, the proofs were substantially more complex than their counterparts for $\lozenge$-closure. Nevertheless, only one property of $\blacklozenge$-closure is slightly weaker than its $\lozenge$-closure counterpart.

The counterpart of comtraces for io-structures has not been fully developed yet, but its foundation has been established. Fishburn's Theorem (Theorem 2) states that each interval order can be represented by an appropriate total order of the interval beginnings and ends. The below fundamental theorem states that each io-structure can be represented by an appropriate partial (not necessarily interval) order of the beginnings and ends.

Theorem 8 (Abraham, Ben-David, Magodor [1]).
A relational structure $S = (X, \prec, \sqsubset)$ is an io-structure iff there exists a partial order $\lhd$ on some Y and two mappings $B, E : X \to Y$ such that $B(X) \cap E(X) = \emptyset$ and for each

1. $B(x) \lhd E(x)$, *2.* $x \prec y \iff E(x) \lhd B(y)$, *3.* $x \sqsubset y \iff B(x) \lhd E(y)$. □

Szpilrajn's Theorem (Theorem 1) allows us to represent each partial order by its total extensions. The combination of these three theorems and Theorem 7 makes it possible to construct "interval traces", a version of Mazurkiewicz traces over an appropriate monoid of sequences of beginnings and ends, and then use "interval traces" to represent io-structures via Theorem 8. This topic is beyond the scope of this paper; however, the properties of $\blacklozenge$-closure are essential tools in this process.

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