Formalizing Randomized Matching Algorithms

Dai Tri Man Lê and Stephen Cook

Department of Computer Science
University of Toronto
Canada

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Two Aspects of Proof Complexity

1. Propositional Proof Complexity *(Pitassi's invited talk)*
   - the lengths of proofs of tautologies in various proof systems
2. Bounded Arithmetic
   - the power of weak formal systems to prove theorems of interest in computer science

- (1) and (2) are related by "propositional translations"
  - a proof in theory $T \rightsquigarrow$ uniform short proofs in propositional system P_T
  - bounded arithmetic = uniform version of propositional proof complexity
- "bounded": induction axioms are restricted to bounded formulas

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Complexity Theory	Bounded Arithmetic
Classify problems according to complexity classes	Classify theorems according to the computational complexity of concepts needed to prove them. "Bounded Reverse Mathematics" [Cook-Nguyen '10]
Separate (or collapse) complexity classes	Separate (or collapse) formal theories for various complexity classes

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Feasible reasoning with VPV

The VPV theory

- associated with complexity class P (polytime)
- universal theory based on Cook's theory *PV* ('75)
- with symbols for all polytime functions and their defining axioms based on Cobham's Theorem ('65).
- Induction on polytime predicates: a derived result via binary search.
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Proofs in *VPV* are **feasibly constructive**.

- Given a proof in *VPV* for the formula $\forall X \exists Y \varphi(X, Y)$, where φ represents a polytime predicate, we can extract a polytime function $F(X)$ and a correctness proof in *VPV* of $\forall X \varphi(X, F(X))$.
- Induction is restricted to polytime "concepts".

Feasible proofs

Polytime algorithms usually have feasible correctness proofs, e.g.,

- the "augmenting-path" algorithm: finding a maximum matching
- the Hungarian algorithm: finding a minimum-weight matching
- ...

(formalized in *VPV*, see the full version on our websites)

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Main Question

How about randomized algorithms and probabilistic reasoning?

"Formalizing Randomized Matching Algorithms"

Two fundamental randomized matching algorithms

1. RNC^2 algorithm for testing if a bipartite graph has a perfect matching (Lovász '79)
2. RNC^2 algorithm for finding a perfect matching of a bipartite graph (Mulmuley-Vazirani-Vazirani '87)

Recall that:

$$\boxed{\begin{array}{c}\text{Log-Space} \subseteq \mathsf{NC}^2 \subseteq \mathsf{P} \\ \mathsf{RNC}^2 \subseteq \mathsf{RP}\end{array}}$$

Important Remark

The two algorithms above also work for general undirected graphs, but we only consider bipartite graphs.

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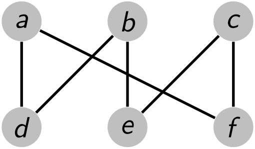
Given a bipartite graph G, decide if G has a perfect matching.



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replace ones with distinct variables $\rightsquigarrow$

$$M_G = \begin{bmatrix} x_{11} & 0 & x_{13} \\ x_{21} & x_{22} & 0 \\ 0 & x_{32} & x_{33} \end{bmatrix}$$

Edmonds' Theorem (provable in VPV)

G has a perfect matching if and only if $\mathrm{Det}(M_G)$ is not identically zero.

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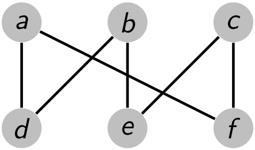

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The usual proof is not feasible since...

it uses the formula $\mathrm{Det}(M) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} M(i, \sigma(i))$, which has $n!$ terms.

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- **Observation:** instance of the polynomial identity testing problem
- $\mathrm{Det}(M_G^{n\times n})$ is a polynomial in n^2 variables x_{ij} with degree at most n.
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Lovász's RNC^2 Algorithm

- Pick n^2 random values r_{ij} from $S = \{0, \ldots, 2n\}$
- If $\mathrm{Det}(M_G)(\vec{r}) = 0$ then YES ($\mathrm{Det}(M_G) \equiv 0$) else NO.

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1. if $\mathrm{Det}(M_G) \equiv 0$, then $\mathrm{Det}(M_G)(\vec{r}) = 0$
2. if $\mathrm{Det}(M_G) \not\equiv 0$, then $\Pr_{\vec{r}\in_R S^{n^2}}\left[\mathrm{Det}(M_G)(\vec{r}) \neq 0\right] \geq 1/2$

((2) follows from the Schwartz-Zippel Lemma)

Obstacle #1 - Talking about probability

- Given a polytime predicate $A(X, R)$,

$$\Pr_{R \in \{0,1\}^n}\big[A(X,R)\big] = \frac{|\{R \in \{0,1\}^n \mid A(X,R)\}|}{2^n}$$

- The function $F(X) := \big|\big\{R \in \{0,1\}^n \mid A(X,R)\big\}\big|$ is in #P.
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Solution [Jeřábek '04]

- We want to show $\Pr_{R\in\{0,1\}^n}\big[A(X,R)\big] \geq s/t$, it suffices to show

$$|\{R \in \{0,1\}^n \mid A(X,R)\}| \cdot t \geq 2^n \cdot s$$

- Key idea: construct in *VPV* a polytime surjection

$$G : \{R \in \{0,1\}^n \mid A(X,R)\} \times [t] \twoheadrightarrow \{0,1\}^n \times [s],$$

where $[m] := \{1, \ldots, m\}$.

Definition (Jeřábek 2004 – modified)

Let $\Gamma, \Delta \subseteq \{0,1\}^n$ be polytime definable sets. Define Γ is "larger" than Δ if there exists a polytime surjective function $F : \Gamma \twoheadrightarrow \Delta$.

A bit of history

A series of papers by Jeřábek (2004–2009) justifying and utilizing the above definition

- A very sophisticated framework
- Based on approximate counting techniques
- Related to the theory of derandomization and pseudorandomness
- Application: formalizing probabilistic complexity classes

The Schwartz-Zippel Lemma

Let $P(X_1, \ldots, X_n)$ be a non-zero polynomial of degree D over a field $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}$. Then

$$\Pr_{\vec{R} \in S^n}\left[P(\vec{R}) = 0\right] \leq \frac{D}{|S|}.$$

Obstacle #2

- The usual proof assumes we can rewrite

$$P(X_1, \ldots, X_n) = \sum_{J=0}^{D} X_1^J \cdot P_J(X_2, \ldots, X_n)$$

- This step is not feasible when P is given as arithmetic circuit or symbolic determinant

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Solution

- Being less ambitious: restrict to the case of Edmonds' polynomials
- Take advantage of the special structure of Edmonds' polynomials

Edmonds' polynomials

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Edmonds' matrix:

Useful observation:

- Each variable x_{ij} appears at most once in M_G.
- From the above example, by the cofactor expansion,

$$\mathrm{Det}(M_G) = -x_{33} \cdot \mathrm{Det}\begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} + \mathrm{Det}\begin{pmatrix} x_{11} & 0 & x_{13} \\ x_{21} & x_{22} & 0 \\ 0 & x_{32} & \mathbf{0} \end{pmatrix}$$

- Thus, we can apply the idea in the original proof.

Theorem (provable in VPV)

Assume the bipartite graph G has a perfect matching.

- Let $S = \{0, \dots, s\}$ be the sample set.
- Let $M_G^{n \times n}$ be the Edmonds' matrix of G.

Then we can construct polytime surjection

$$F : [n] \times S^{n^2-1} \twoheadrightarrow \{\vec{r} \in S^{n^2} \mid \mathrm{Det}(M_G)(\vec{r}) = 0\}.$$

- The degree of the polynomial $\mathrm{Det}(M_G)$ is at most n.
- The surjection F witnesses that

$$\Pr_{\vec{r} \in S^{n^2}}\left[\mathrm{Det}(M_G)(\vec{r}) = 0\right] = \frac{\left|\left\{\vec{r} \in S^{n^2} \mid \mathrm{Det}(A)(\vec{r}) = 0\right\}\right|}{s^{n^2}} \leq \frac{n}{s}$$

The Mulmuley-Vazirani-Vazirani Algorithm

- RNC^2 algorithm for finding a perfect matching of a bipartite graph
- Key idea: reduce to the problem of finding a **unique** min-weight perfect matching using **the isolating lemma**.

Obstacle

The isolating lemma seems too general to give a feasible proof.

Solution

Consider a specialized version of the isolating lemma.

Lemma

Given a bipartite graph G. Assume the family $\mathcal{F}$ of all perfect matchings of G is nonempty. If we assign random weights to the edges, then

$$\Pr\Big[\textit{the min-weight perfect matching is unique}\Big] \textit{ is high.}$$

Summary

Main motivation

Feasible proofs for randomized algorithms and probabilistic reasoning:
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Feasible proofs for randomized algorithms and probabilistic reasoning:
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We demonstrate the techniques through two randomized algorithms:

1. RNC^2 algorithm for testing if a bipartite graph has a perfect matching [Lovász '79]
   - the Schwartz-Zippel Lemma for Edmonds' polynomials
2. RNC^2 algorithm for finding a perfect matching of a bipartite graph [Mulmuley-Vazirani-Vazirani '87]
   - a specialized version of the isolating lemma for bipartite matchings.

Take advantage of special linear-algebraic properties of
Edmonds' matrices and Edmonds' polynomials

Open questions

1. Can we prove in VPV more general version of the Schwartz-Zippel lemma? *(We only considered Edmonds' polynomials.)*
2. Can we do better than VPV, e.g., VNC^2 [Cook & Nguyen '10]?

Open problems and future work

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Future work

1. How about RNC^2 matching algorithms for undirected graphs?
   - Use properties of the **pfaffian**
   - Need to generalize results from [Soltys '01] [Soltys & Cook '02] (with Cook and Fontes)
2. Use Jeřábek's techniques to formalize constructive aspects of fundamental theorems that require probabilistic reasoning.
   - Cryptography: the Goldreich-Levin Theorem, construction of pseudorandom generator from one-way functions, etc. (with George)
   - Moser-Tados constructive proof of Lovász Local Lemma (with Filmus)