Fall 2001	CSC2423	Finite model theory and descriptive complexity

Assignment 1

Exercises (due date: November 2).

(a) Prove that 2-colorability and 3-colorability of finite graphs are not FO-definable.

An m-ary query is FO-expressible on a class of structures $\mathcal{C}$ if there exists an FO-formula $\varphi(\vec{x})$ such that, for any $\mathcal{A} \in \mathcal{C}$, $Q(\mathcal{A}) = \{\vec{a} \in A^m \mid \mathcal{A} \models \varphi(\vec{a})\}$.

Prove that the following are not FO-expressible over (rooted, directed) trees:

(b) being balanced – complete the proof based on Hanf's theorem, which we sketched in class;

(c) same generation: this is a binary query; (x, y) are in the same generation if they are at the same distance from any of their common ancestors;

(d) same generation again, but this time over classes of structures $\mathcal{A} = (A, E_1, E_2)$ where E_1 is a tree, and E_2 is the transitive closure of E_1. (If you came up with a very quick proof of (c), you have to do a bit of work at this point; if you had a more elaborate proof of (c), you're probably done now.)

(e) Provide a complete proof of the game characterization of MSO.

Problems.

(1) Two problems related to BNDP:

1, a) Prove that every local m-ary query has the BNDP, for $m > 2$ (we saw the case of $m = 2$ in class).

1, b) Consider a statement more symmetric than BNDP: a binary query Q has S(ymmetric)BNDP if there exists a function f_Q such that for any graph G realizing at most k different degrees, $Q(G)$ realizes at most $f_Q(k)$ different degrees.

Do all FO-definable queries on finite graphs have the SBNDP? If yes, prove it, if not, give a counterexample.

(2) We've seen that in FO($\mathbf{C}$), BIT and $+, \cdot$ are interdefinable. Prove the same for FO($<$). Next, show how to express $|X| \leq \log_2 |Y|$ in FO($<$, BIT).

Note: FO(BIT, $<$) = FO($+, \cdot, <$) is a "folklore" statement, but no satisfactory proof can be found in the (accessible) literature.

Challenges.

(α) Give an example of an order-invariant FO($<$) query which is not definable in $\mathcal{L}^*_{\infty\omega}(\mathbf{C})$.

(β) Prove that every invariant FO($<, +$) query is Gaifman-local (you may find it conventient to start with the case of one free variable).