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On Channel Access Delay of CSMA Policies in Wireless Networks with Primary Interference Constraints

Mahdi Lotfinezhad and Peter Marbach
Department of Computer Science, University of Toronto
Email: {mahdi,marbach}@cs.toronto.edu

***Abstract*—CSMA policies are examples of simple distributed scheduling algorithms in wireless networks. In this paper, we study the delay properties of CSMA in the limit of large networks. We first define an ideal network under which CSMA becomes memoryless in that the delay to access the channel becomes an exponential r.v. independent of the past. We then show that, in the limit of large bipartite graphs with primary interference constraints, simple CSMA policies become memoryless.**

I. INTRODUCTION

CSMA policies are simple distributed policies, which play an essential role in many of the current standards. It has been shown that CSMA policies are throughput optimal for multihop networks under fairly general interference models [1]. In the case of primary interference model [6], [7], more explicit results are provided in [2] characterizing the rate region of CSMA policies in the limit of large networks with many small flows. One fundamental question is how the packet delay behaves as the size of the network increases or the boundary of capacity region is approached. This delay is closely related to the delay to access the channel due to the contention among nodes. Once the *access delay* is properly characterized, one can also obtain bounds for the queueing delay. Literature [3] [4] suggests that in general CSMA may lock into a schedule for a long time, making the access delay unacceptably large. This implies that CSMA has schduling memory that causes short-term unfairness.

Ideally, we would like to have access delays as renewal intervals that have exponential distribution. Motivated by this, we define a scheduling policy π to be *memoryless* for a given network and a given arrival rate vector if it satisfies the following condition:

- Under the policy π, for any link, the delays to access the channel (to transmit one packet) scaled by the arrival rate of that link are 1) i.i.d. random variables, and 2) are exponentially distributed with unit mean.

This property ensures that the access delays are not correlated in time and also ensures short-term fairness.

Our goal is to examine nontrivial cases under which CSMA policies become memoryless. As the first step, we define an ideal network under which CSMA becomes memoryless. We use the intuition gained to provide an upper bound for the average access delay in the limit of large networks with many small flows as studied in [2]. We then consider networks that can be represnted by bipartite graphs. We show that under some conditions, one can use the results from the mean-field theory [5] to explicitly predict the behaviour of CSMA on these networks. Specifically, we can accurately predict the fraction of idle nodes over any finite interval of time. We use this prediction and show that in the limit when the size of the bipartite networks grows to infinity, CSMA becomes memoryless.

II. SYSTEM MODEL

We consider a wireless network $(\mathcal{N}, \mathcal{L})$ composed of a set $\mathcal{N}$ of nodes with cardinality N, and a set $\mathcal{L}$ of directed links with cardinality L. A directed link $(i, j) \in \mathcal{L}$ indicates that node i is able to send data packets to node j. We assume that the rate of transmission is the same for all links. For simplicity, we assume the system is time-slotted and each timeslot has duration β time units. Further, we assume that packet lengths are geometrically distributed with unit mean, and we rescale time such that the time it takes to transmit one packet is on the average equal to one time unit. We define $\mathcal{N}_i$ to be the set of all neighbours of node i, i.e., $\mathcal{N}_i = \{j; (i, j) \in \mathcal{L} \text{ or } (j, i) \in \mathcal{L}\}$. We also define $\mathcal{L}_i = \{l; l = (i, j) \in \mathcal{L}\}$ to be set of all outgoing links from node i.

Interference Model: We focus on networks under the well-known *primary interference*, or *node exclusive interference*, model [6], [7]. In this model, a packet transmission over link (i, j) is successful if only if within the transmission duration, there exists no other activity over any other link (m, n) which shares a node with (i, j). The primary interference model applies, for example, to wireless systems where multiple frequencies/codes are available (using FDMA or CDMA) to avoid interference, but each node has only a single transceiver and hence can only send to or receive from one other node at any time.

Traffic Model: We characterize the network traffic by a rate vector $\lambda := \{\lambda_r\}_{r\in\mathcal{R}}$ where $\mathcal{R}$ is the set of routes used by the traffic, and λ_r, $\lambda_r \geq 0$, is the mean rate in packets per unit time along route $r \in \mathcal{R}$.

For a given route $r \in \mathcal{R}$, let s_r be its source node and d_r be its destination node, and let

$$\mathcal{R}_r = \{(s_r, i), (i, j), \cdots, (v, w), (w, d_r)\} \subset \mathcal{L}$$

be the set of links traversed by the route. We allow several routes to be defined for a given source and destination pair

(s,d), $s,d \in \mathcal{N}$.

Given the rate vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$, we let

$$\lambda_{(i,j)} = \sum_{r:(i,j)\in\mathcal{R}_r} \lambda_r, \qquad (i,j)\in\mathcal{L}, \tag{1}$$

be the mean packet arrival rate to link (i,j). Similarly, we let

$$\Lambda_i = \sum_{j\in\mathcal{N}_i} \left[\lambda_{(i,j)} + \lambda_{(j,i)}\right], \qquad i\in\mathcal{N}.$$

be the mean packet arrival rate to node $i\in\mathcal{N}$. Introduction of $\lambda_{(i,j)}$ allows us to focus on single hop packet transmissions and on the delay of accessing the channel for any given link.

III. Capacity Region

Consider a fixed network $(\mathcal{N},\mathcal{L})$ with traffic vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$. A scheduling *policy* π then defines the rules that are used to schedule packet transmissions on each link $(i,j)\in\mathcal{L}$. We define the link service rates as a function of the rate vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$.

Definition 1 (Service Rate)**.** *Consider a fixed network $(\mathcal{N},\mathcal{L})$. The link service rate $\mu^{\pi}_{(i,j)}(\lambda)$, $(i,j)\in\mathcal{L}$, of policy π for the traffic vector $\lambda=\{\lambda_r\}_{r\in\mathcal{R}}$ is the fraction of time node i spends successfully transmitting packets on link (i,j) under π and λ, i.e. the fraction of time node i sends packets over link (i,j) that do not experience a collision.*

Let $\mathcal{P}$ be the class of all policies π that have well-defined link service rates. We then define network stability as follows.

Definition 2 (Stability)**.** *For a given network $(\mathcal{N},\mathcal{L})$, let $\mu^{\pi}(\lambda) = \{\mu^{\pi}_{(i,j)}(\lambda)\}_{(i,j)\in\mathcal{L}}$ the link service rates of policy π, $\pi\in\mathcal{P}$, for the rate vector $\lambda=\{\lambda_r\}_{r\in\mathcal{R}}$. We say that policy π stabilizes the network for λ if $\lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda)$, $(i,j)\in\mathcal{L}$.*

Based on this stability criteria, the capacity region of a network $(\mathcal{N},\mathcal{L})$ is then defined as follows.

Definition 3 (Capacity Region)**.** *For a given a network $(\mathcal{N},\mathcal{L})$, the capacity region $\mathcal{C}$ is equal to the set of all traffic vectors $\lambda=\{\lambda_r\}_{r\in\mathcal{R}}$ such that there exists a policy $\pi\in\mathcal{P}$ that stabilizes the network for λ, i.e. we have*

$$\mathcal{C} = \{\lambda \geq 0 : \exists\pi\in\mathcal{P} \text{ with } \lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda),\ \forall(i,j)\in\mathcal{L}\}.$$

IV. CSMA Policies and Their Rate Region

In section, we first define a CSMA policy and then elaborate on their achievable rate region.

A. CSMA Policies

A CSMA policy is given by a transmission attempt probability vector $\mathbf{p} = (p_{(i,j)})_{(i,j)\in\mathcal{L}} \in [0,1]^L$ and a sensing period (or idle period) $\beta > 0$. The policy works as follows: each node, say i, senses the activity on its outgoing links $l\in\mathcal{L}_i$. We say that i has sensed link $(i,j)\in\mathcal{L}_i$ to be idle for a duration of an idle period β if for the duration of β time units we have that (a) node i has not sent or received a packet and (b) node i has sensed that node j has not sent or received a packet. If node i has sensed link $(i,j)\in\mathcal{L}_i$ to be idle for a duration of an idle period β, then i starts a transmission of a single packet on link (i,j) with probability $p_{(i,j)}$, independent of all other events in the network[1] . If node i does not start a packet transmission, then link (i,j) has to remain idle for another period of β time units before i again has the chance to start a packet transmission. Thus, the epochs at which node i has the chance to transmit a packet on link (i,j) are separated by periods of length β during which link (i,j) is idle, and the probability that i starts a transmission on link (i,j) after the link has been idle for β time units is equal to $p_{(i,j)}$,

Recall that we assume a time-slotted system, where each timeslot has a duration of β. Further, we assume that when packets are not available *dummy* packets are transmitted over a link. We distinguish between a link being busy and active. Specifically, we define:

Definition 4 (Active Link)**.** *A link (i,j) is defined to be active at time t, if node i is transmitting to node j at time t.*

Definition 5 (Busy Link)**.** *A link (i,j) is defined to be busy at time t, if at least one of nodes i and j are transmitting to one of their neighbouring nodes.*

B. Rate Region of CSMA Policies with Primary Interference

Here, we provide key observations provided in [2] regarding the rate region and throughput-optimality of CSMA policies in the limit of large networks. We first give a definition.

Definition 6 (Achievable Rate Region of CSMA Policies)**.** *For a given network $(\mathcal{N},\mathcal{L})$ and a given sensing period β, the* achievable rate region of CSMA policies *is given by the set of rate vectors $\lambda=\{\lambda_r\}_{r\in\mathcal{R}}$ for which there exists a CSMA policy $\mathbf{p}$ that stabilizes the network for λ, i.e. we have that $\lambda_{(i,j)} < \mu_{(i,j)}(\mathbf{p})$, $(i,j)\in\mathcal{L}$.*

For given network $(\mathcal{N},\mathcal{L})$, let $\Gamma(\beta)$ be the achievable rate region of CSMA policies when the duration of an idle period is equal to β. In [2], it is shown that

$$\lim_{\beta\downarrow 0}\Gamma(\beta) = \{\lambda \geq \mathbf{0} : \Lambda_i < 1, \text{ for all } i\in\mathcal{N}\} \tag{2}$$

for *large networks with many small flows and small sensing period*. Since it is impossible for any policy to stabilize the network with a node i requiring $\Lambda_i \geq 1$, this result states $\lim_{\beta\downarrow 0}\Gamma(\beta) = \mathcal{C}$, and thus, CSMA policies are (asymptotically) throughput-optimal.

To formally define these networks, consider a sequence $\{(\mathcal{N}^{(N)},\mathcal{L}^{(N)})\}$ of networks for which the number of nodes N increases to infinity. Let $\{\lambda^{(N)}\}$ be the corresponding sequence of arrival rate vectors for all links. A sequence of networks with many small flows and small sensing period is the one for which the following hold:

- $\lim_{N\to\infty} N\beta^{(N)} = 0$.
- $\limsup_{N\to\infty}\left(\max_{(i,j)\in\mathcal{L}^{(N)}}\lambda^{(N)}_{(i,j)}\right) = 0$.

[1]To avoid self-interference, where a node may transmit over two outgoing links, nodes first decide to transmit with probability $\sigma = \sum_{\{(i,j):(i,j) \text{ idle}\}} p_{(i,j)}$. Once they decide to transmit, they choose (only) one link, say (i,j_1), with probability $\sigma^{-1}p_{(i,j_1)}$.

The constructive proof in [2] shows that to stabilize $\lambda^{(N)}$ for these networks in the limit of large N, one can choose

$$p_{(i,j)}^{(N)} = \frac{\lambda_{(i,j)}^{(N)}}{\rho_i^{(N)}\rho_j^{(N)}}\beta e^{2G^{+}(\beta^{(N)})}, \tag{3}$$

where $G^{+}(\beta) = \sqrt{2\beta}$ for $\beta > 0$, and $\rho_i^{(N)}$ is the probability that node i is idle. This probability is the solution of the following fixed point equations:

$$\rho_i^{(N)} = \frac{\beta^{(N)}}{\beta^{(N)} + 1 - e^{G_i^{(N)}}},\ i = 1, \cdots, N, \tag{4}$$

$$G_i = \sum_{j\in\mathcal{N}_i} (p_{(i,j)}^{(N)} + p_{(j,i)}^{(N)})\rho_j^{(N)},\ i = 1, \cdots, N. \tag{5}$$

Note that these equations are similar to those of classical single-cell networks [2].

V. An Ideal Picture

In this section, we define an *ideal network* and discuss its access delay properties. Consider link (i, j). We define an ideal network to be the one in which for any link $l = (i, j)$ the following hold:

- When link l is not active, and at least one of the nodes i or j are active, the idle and active periods of node i (node j) are geometrically distributed with mean $\frac{1}{\kappa_i}$ and one, respectively, independent of activities of nodes j (node i).
- When link l is idle, in the next timeslot, with probability $p_{(i,j)} = r_{(i,j)}\beta$, the link becomes active, otherwise nodes i or j independently of each other become active with probability $\kappa_i\beta$ or $\kappa_j\beta$, respectively.

Under the above ideal hypothesis, the idle and active periods of link (i, j) can be characterized by a simple Markov chain with state $S(n)$ at timeslot n. The state space of this Markov chain can be defined as $\mathcal{S} = \{S_k, k = 1, \cdots, 5\} = \{(s_i, s_j)\} \cup \{(l)\}$, where $s_i = 0$ means node i is idle, and $s_i = 1$ means it is active. The state (l) means that link l is active (transmitting). Fig. 1 shows how the an ideal network will look like from the view point of link l, in terms of a Markov chain with *transition rates* in the limit of $\beta \to 0$.

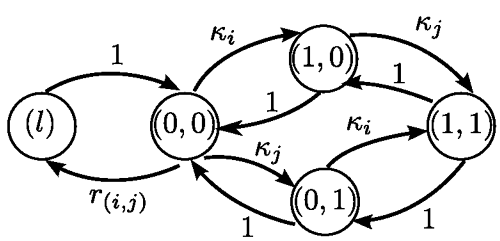

Fig. 1. Markov chain from the view point of link (i, j). We have shown *transition rates* in the limit of $\beta \to 0$.

To simplify the analysis, in the rest of this section, we consider the limit of $\beta \to 0$. In this limit, the Markov chain becomes irreducable, and we have

$$\lim_{\beta\to 0} \frac{\pi_{(0,0)}}{(1 + r_{(i,j)} + \kappa_i + \kappa_j + \kappa_i\kappa_j)^{-1}} = 1.$$

Since $\pi_l = r_{(i,j)}\pi_{(0,0)}$, we also have that

$$\lim_{\beta\to 0} \frac{\pi_l}{r_{(i,j)}(1 + r_{(i,j)} + \kappa_i + \kappa_j + \kappa_i\kappa_j)^{-1}} = 1.$$

We see that by choosing

$$r_{(i,j)} = \frac{\lambda_{(i,j)}}{1 - \lambda_{(i,j)}}(1 + \kappa_i)(1 + \kappa_j), \tag{6}$$

asymptotically $\frac{\pi_l}{\lambda_{(i,j)}} = 1$. Hence, if the ideal hypothesis holds, it becomes fairly easy to choose the correct transmission probabilities. Based on the above, when κ_i and κ_j are large and $\lambda_{(i,j)}$ is small, almost every $\kappa_i\kappa_j$ timeslots, link l finds an opportunity to transmit with probability $r_{(i,j)} = \kappa_i\kappa_j\lambda_{(i,j)}$ so that on the average every $\frac{1}{\lambda_{(i,j)}}$ timeslot, link l transmits for an average of one unit time. One can immediately see that the fraction of time that node i is idle, ρ_i, satisfies

$$\lim_{\lambda_{(i,j)},\beta\to 0} \rho_i = \frac{1}{1 + \kappa_i}. \tag{7}$$

Hence, considering (3) and (6) , we see that the ideal network assumption leads to the same CSMA policy as the one used for the networks with many small flows.

Next, we obtain a bound on the second moment of the time between consequitive transmissions of link l. For simplicity, in the rest, we assume $\kappa_i = \kappa_j = \kappa > 0.5$. First, suppose that the chain is at the state $(1, 1)$. Consider the next three transitions in each of which the state *changes*. In the first transition, with probability one, the state changes to either $(0, 1)$ or $(1, 0)$. In the next transition, with probability $\frac{1}{1+\kappa}$ the state changes to $(0, 0)$. Once in $(0, 0)$, in the next tranistion, with probability

$$p_s = \frac{(1 + \kappa)^2\lambda_{(i,j)}}{2\kappa(1 - \lambda_{(i,j)}) + (1 + \kappa)^2\lambda_{(i,j)}}, \tag{8}$$

the state changes to (l). Hence, assuming three consequtive transitions to new states, with probability

$$\frac{1}{(1 + \kappa)} \frac{(1 + \kappa)^2\lambda_{(i,j)}}{(2\kappa(1 - \lambda_{(i,j)}) + (1 + \kappa)^2\lambda_{(i,j)})} \tag{9}$$

link l becomes active. The expected time for the above sequence of events is $0.5 + \frac{1}{1+\kappa} + \frac{(1-\lambda_{(i,j)})}{2\kappa(1-\lambda_{(i,j)})+\lambda_{(i,j)}(1+\kappa)^2}$. This means that since $\kappa > 0.5$, for some fixed $0 < C < 1$, within 0.5 unit time, with probability at least

$$C\frac{\lambda_{(i,j)}}{2 + (\kappa + 1)\lambda_{(i,j)}}, \tag{10}$$

the link l becomes active. One can obtain a similar lower-bound if the intial state is any other state. We choose C to be the minimum constant resulting from different intial states.

Define τ_l, as the access delay, to be the time until the next packet transmission by link l given that link l is not currently active. By the above discussion, we know that this time has exponential decay. Using (10), we can show that for a fixed κ,

$$\lim_{\lambda_{(i,j)}\to 0} \frac{\mathbb{E}[\tau_l^2]}{(C\lambda_{(i,j)})^{-2}} \leq 1. \tag{11}$$

But we have that $\mathbb{E}[\tau_l^2] \geq \mathbb{E}[\tau_l]^2 = (\lambda_{(i,j)})^{-2}$. Therefore,

$$\mathbb{E}[\tau_l^2] = \mathbf{\Theta}\Big(\frac{1}{\lambda_{(i,j)}^2}\Big), \text{ as } \lambda_{(i,j)} \to 0. \tag{12}$$

This suggests that the access delay τ_l should be exponentially distributed with mean $\lambda_{(i,j)}$. We have the following theorem:

Theorem 1. *Let $\tau_l^s = \lambda_{(i,j)}\tau_l$ be the scaled version of the access delay τ_l. Under the ideal network assumption, independent of the past*

$$\lim_{\lambda_{(i,j)}\to 0}\lim_{\beta\to 0}\tau_l^s \xrightarrow{\mathcal{D}} X,$$

where X is an exponential r.v. with unit mean, and convergence is in distribution.

This theorem essentially states that, due to the Markovian nature of the system, the scaled access delays are i.i.d. and exponentially distributed with unit mean. Hence, CSMA is memoryless on the ideal networks with small flows.

Proof of Theorem 1: The main idea here is that whenever the state returns to $(0,0)$, upon transition to a new state, with probability p_s given in (8), link l becomes active for average duration of one unit time. If $\lambda_{(i,j)}$ is suffieintly small, then many returns to $(0,0)$ must be made before link l transmits. These many attempts remove the uncertainty of the length of return times to $(0,0)$. In such a case, it seems as if link l attempts to transmit with probability p_s in regular time intervals; hence, we expect a geometric distribution. In the limit, when p_s is small, by proper scaling, we obtain an exponential distribution, which is the claim of the theorem.

Define a *transition epoch* to be the timeslot t_e, where the state changes from the previous timeslot such that $S(t_e - 1) = (0,0)$ and $S(t_e) \neq (0,0)$. Recall that every timeslot has duration β. Define $\hat{\tau}_{(0,0)}$ to be the time between two consequtive transition epochs $t_e^{(1)}$ and $t_e^{(2)}$ given that the state never becomes (l) between the epochs. Let

$$\bar{\tau} = \mathbb{E}[\hat{\tau}_{(0,0)}].$$

We have that

$$\lim_{\lambda_{(i,j)}\to 0}\lim_{\beta\to 0}\bar{\tau} = \frac{(1+\kappa)^2}{2\kappa}. \tag{13}$$

In addition,

$$\lim_{\lambda_{(i,j)}\to 0}\lim_{\beta\to 0}\frac{p_s}{\lambda_{(i,j)}} = \frac{(1+\kappa)^2}{2\kappa}. \tag{14}$$

Consider a point t on the real time axis. By this point, there must be $\lfloor \frac{t}{\beta} \rfloor$ timeslots. Suppose timeslot zero is the first transition epoch. Define a failuare as an event where once in $(0,0)$, the next *new* state is not (l) and link l does not transmit. Once in $(0,0)$ this event happens with probability $(1-p_s)$. Let the event E_j be defined as

$$E_j = \{\text{There are } j \text{ failures upto and including time } t \text{ and the } j+1\text{th transition epoch occurs after time } t.\} \tag{15}$$

We note that given a failuare event corresponding to a transition epoch, the time from begining of the epoch to the next epoch, is a r.v., which we already defined as $\hat{\tau}_{(0,0)}$. Given that there are j consequitive failuares, define $\{\hat{\tau}_{(0,0)}^{(k)},\ k = 1,\cdots,j\}$ to be sequence of $\hat{\tau}_{(0,0)}$'s. We note that given E_j, the corresponding r.v.'s $\hat{\tau}_{(0,0)}^{(k)}$, $k = 1,\cdots,j$, are independent of each other since the system is Markovian.

Using the above observations and definitions, for the probability that link l does not transmit by time t, we have

$$\begin{aligned} P(\tau_l > t) = \sum_{j=1}^{\infty} P(E_j) = & \\ & \sum_{j > \frac{t}{\bar{\tau}}(1+\epsilon)+1} P\big(E_j'\big)(1-p_s)^j && (16)\\ & + \sum_{j < \frac{t}{\bar{\tau}}(1-\epsilon)-1} P\big(E_j'\big)(1-p_s)^j && (17)\\ & + \sum_{|j-\frac{t}{\bar{\tau}}| < \epsilon\frac{t}{\bar{\tau}}+1} P\big(E_j'\big)(1-p_s)^j, && (18)\end{aligned}$$

where $\epsilon > 0$ is chosen such that $\epsilon\frac{t}{\bar{\tau}}$ is an integer, and

$$E_j' = \{\text{Event that } \sum_{k=1}^{j}\hat{\tau}_{(0,0)}^{(k)} > t.\}.$$

We treat each summation in sequence. The event $j > \frac{t}{\bar{\tau}}(1+\epsilon)+1$ implies that with probability one

$$\sum_{k=1}^{j-1}\hat{\tau}_{(0,0)}^{(k)} < t, \tag{19}$$

which can be written as

$$\frac{\sum_{k=1}^{j-1}\hat{\tau}_{(0,0)}^{(k)}}{j-1} - \bar{\tau} < \frac{t}{j-1} - \bar{\tau} < -\bar{\tau}\frac{\epsilon}{1+\epsilon}. \tag{20}$$

As mentioned earlier, $\hat{\tau}_{(0,0)}^{(k)}$'s are i.i.d., and hence, by large deviation results [8], we have that the probability of (20) is upperbounded by $\delta(\epsilon)^{j-1}$, where $0 < \delta(\epsilon) < 1$. Hence

$$P\Big(E_j', j > \frac{t}{\bar{\tau}}(1+\epsilon)+1\Big) < \delta(\epsilon)^{j-1} < \delta(\epsilon)^{\frac{t}{\bar{\tau}}(1+\epsilon)}. \tag{21}$$

To examine the case where $j < \frac{t}{\bar{\tau}}(1-\epsilon)-1$, note that this event requires that

$$\sum_{k=1}^{j}\hat{\tau}_{(0,0)}^{(k)} > t. \tag{22}$$

We consider two subcases. First, suppose $j > \frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon)$. Hence, we must have that

$$\frac{\sum_{k=1}^{j}\hat{\tau}_{(0,0)}^{(k)}}{j} - \bar{\tau} > \frac{t}{j} - \bar{\tau} > \bar{\tau}\frac{\epsilon}{1-\epsilon} > \bar{\tau}\frac{\epsilon}{1+\epsilon}. \tag{23}$$

By large deviations for this case, we also have that

$$\begin{aligned} P\Big(E_j',\ \frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon) < j < \frac{t}{\bar{\tau}}(1-\epsilon)-1\Big) &< \delta(\epsilon)^j \\ &< \delta(\epsilon)^{\frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon)}. \end{aligned} \tag{24}$$

For the second subcase where $j \leq \frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon)$, by positivity and independence of r.v.'s, from the last inequality, it simply follows that

$$P\Big(E_j', j < \frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon)\Big) < \delta(\epsilon)^{\frac{\sqrt{t}}{\bar{\tau}}(1-\epsilon)}. \tag{25}$$

We now consider the third summation at (18). Let x denote the scaled version of t so that $x = \lambda_{(i,j)} t$. We have

$$\sum_{|j-\frac{t}{\bar\tau}|<\epsilon\frac{t}{\bar\tau}+1} P\big(E'_j\big)(1-p_s)^j$$
$$= (1-p_s)^{\frac{t}{\bar\tau}} \sum_{|j-\frac{t}{\bar\tau}|<\epsilon\frac{t}{\bar\tau}+1} (1-p_s)^{j-\frac{t}{\bar\tau}} P(E'_j)$$
$$= (1-p_s)^{\frac{x}{\lambda_{(i,j)}\bar\tau}} \sum (1-p_s)^{j-\frac{x}{\lambda_{(i,j)}\bar\tau}} P(E'_j). \quad (26)$$

Noting that by (13) and (14)

$$\lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} (1-p_s)^{\frac{x}{\lambda_{(i,j)}\bar\tau}} =$$
$$\lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} (1-p_s)^{\frac{x}{-p_s}\frac{p_s}{\lambda_{(i,j)}\bar\tau}} = e^{-x}, \quad (27)$$

and that, for j with $|j-\frac{t}{\bar\tau}| < \epsilon\frac{t}{\bar\tau}+1$, we have

$$(1-p_s)^{\frac{\epsilon x}{\lambda_{(i,j)}\bar\tau}} \le (1-p_s)^{j-\frac{x}{\lambda_{(i,j)}\bar\tau}} \le (1-p_s)^{-\frac{\epsilon x}{\lambda_{(i,j)}\bar\tau}}, \quad (28)$$

we can multiply all sides of (28) by $P(E'_j)$, take the summation and the limit, and use (26) and (27) to obtain

$$e^{-(1+\epsilon)x} \lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} \sum_{|j-\frac{t}{\bar\tau}|<\epsilon\frac{t}{\bar\tau}+1} P\big(E'_j\big)$$
$$\le \lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} \sum_{|j-\frac{t}{\bar\tau}|<\epsilon\frac{t}{\bar\tau}+1} P\big(E'_j\big)(1-p_s)^j$$
$$\le e^{-(1-\epsilon)x} \lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} \sum_{|j-\frac{t}{\bar\tau}|<\epsilon\frac{t}{\bar\tau}+1} P\big(E'_j\big) \quad (29)$$

From (21), (24), and (25), we have that for some $0 < \delta'(\epsilon) < 1$

$$P\Big(|j-\frac{\sqrt{t}}{\bar\tau}| > \epsilon\frac{\sqrt{t}}{\bar\tau}+1\Big) < 3\delta'(\epsilon)^{\sqrt{t}} = \delta'(\epsilon)^{\frac{\sqrt{x}}{\lambda_{(i,j)}}}. \quad (30)$$

Letting $\lambda_{(i,j)} \to 0$

$$\lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} P\Big(|j-\frac{t}{\bar\tau}| > \epsilon\frac{t}{\bar\tau}+1\Big) = 0. \quad (31)$$

Since by definition $\tau_l^s = \lambda_{(i,j)}\tau_l$, it immediately follows from (29) and (31) that

$$e^{-(1+\epsilon)x} \le \lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} P(\tau_l^s > x) \le e^{-(1-\epsilon)x}.$$

Since the choice for $\epsilon > 0$ is arbitrary, we have

$$\lim_{\lambda_{(i,j)}\to 0} \lim_{\beta\to 0} P(\tau_l^s > x) = e^{-x}.$$

The proof is complete by noting that we can obtain the same limit starting from all states $S \neq (l)$. ■

VI. A General Bound for Access Delay

In this section, we obtain a general bound for the access delay in networks with many small flows described in Section IV-B. For simplicity, we drop the dependence on N. Here, we do not assume the ideal hypothesis used in the previous section. To simplify the analysis, suppose $\Lambda_i = \Lambda$ for all $i \in \mathcal{N}$. Based on (2), this assumption implies that the node-wise distance to the boundary of capacity region is $1-\Lambda$. From (4) and (5), it follows that

$$\lim_{\beta\to 0} \rho_i = 1-\Lambda,\ i \in \mathcal{N}. \quad (32)$$

We redefine κ in this section so that similar to (7), we have

$$\lim_{\beta\to 0} \rho_i = \frac{1}{1+\kappa},\ i \in \mathcal{N}. \quad (33)$$

Note that by (3)

$$\lim_{\beta\to 0} \frac{p_{(i,j)}}{\beta} = \lambda_{(i,j)}(1+\kappa)^2. \quad (34)$$

Using the above definitions, for the maximum probabilty rate that node i becomes active, we obtain

$$G_{max} = \lim_{\beta\to 0} \frac{1}{\beta} \sum_j (p_{(i,j)} + p_{(j,i)}) = \kappa(\kappa+1) \quad (35)$$

The key to obtain the bound lies in the fact that packet lengths accross all nodes are i.i.d. and geometrically distributed. This implies that while the activities of node i may be correlated to those of node j, given that for instance, node i is transmitting to one other node $k \neq j$, its packet transmission is geometrically distributed with unit mean, independent of the current state of node j.

Considering the above, and that independent of the past, $G_{max}\beta$ is maximum attempt probability for any node, one can construct a worst case Markov chain similar to the one in the previous section and show that, independent of the past, the average delay for the next access to the channel is $O(\frac{(1+\kappa)^2}{\lambda_{(i,j)}})$ as $\lambda_{(i,j)} \to 0$. Since by (32) and (33), $(1+\kappa) = (1-\Lambda)^{-1}$, we have the following:

Theorem 2. *In the limit of large networks with many small flows and small sensing period β, as defined in Section IV-B, regardless of the past history, the expected time for the next access to the channel is $O(\frac{1}{(1-\Lambda)^2}\frac{1}{\lambda_{(i,j)}})$.*

VII. Large Bipartite Graphs as Ideal Networks

In this section, we show that ideal networks as defined in Section V are indeed feasible by considering the limit of large bipartite graphs. Consider a sequence of networks $\{(\mathcal{N}^{(N)}, \mathcal{L}^{(N)}), N \in \mathbb{N}\}$, where each network can be represented as a bipartite graph with a set of N sender nodes $\mathcal{N}_S = \{1, \cdots, N\}$ and a set of N receiver nodes $\mathcal{N}_R = \{N+1, \cdots, 2N\}$. For the Nth network, we assume the sensing period is $\beta^{(N)}$ such that

$$\lim_{N\to\infty} N^2 \beta^{(N)} = 0. \quad (36)$$

We also assume a symmetric CSMA policy $\mathbf{p}^{(N)} = \{p_{(i,j)}^{(N)}\}$ on each network where

$$p_{(i,j)}^{(N)} = \frac{\kappa^2}{N}\beta^{(N)},\ i \in \mathcal{N}_S,\ j \in \mathcal{N}_R. \quad (37)$$

Let $Y^{(N)}(n) = (Y_{(i,j)}^{(N)}(n), i \in \mathcal{N}_S,\ j \in \mathcal{N}_R)$ be the vector of link states at timeslot n, where $Y_{(i,j)}^{(N)}(n) = 1$ indicates that link (i,j) is transmitting, otherwise $Y_{(i,j)}^{(N)}(n) = 0$. Similarly, let $Y_S^{(N)}(n) = (Y_{(i)}^{(N)}(n), 1 \in \mathcal{N}_S)$ and $Y_R^{(N)}(n) = (Y_{(j)}^{(N)}(n), j \in \mathcal{N}_R)$ be the vectors of sender and reciver nodes states at timeslot n, respectively. The vector $Y^{(N)}(n)$ evolves according to a Markov chain.

A. Mean Field Approximation

Here, we show that using mean field theory [5], one can use a simple ODE to characterize the CSMA behaviour on large bipartite networks. First, we note the following two observations:

- Given that there are no collisions at timeslot n, the total number of active links is equal to the number of transmitting nodes, which is equal the number of receiving nodes.
- Given that there are no collisions at timeslot n, the state of links or nodes at timeslot $n+1$ only depends on $Y_S^{(N)}(n)$.

The above means that if we could isolate the effects of collisions, then the total number of idle or active sender nodes could also evolve as a simple Markov chain. We therefore consider a modified system where we add to the state of nodes the *collision state* c. Once the state of a node becomes c, its state never changes. Upon reaching the state c, a sender node never attempts to transmit afterwards, and a receiver node stays busy forever so that it cannot receive packets from any sender node. Let $\tilde{Y}^{(N)}(n)$, $\tilde{Y}_S^{(N)}(n)$, and $\tilde{Y}_R^{(N)}(n)$ be vectors showing the states of links, sender nodes, and receiver nodes in the modified system, respectively.

The modified system works as follows. Starting from an initial state of no collisions, it operates, the same as the original system until the first collision event happens. Upon this event, the state of sender nodes involved in collision changes to c, and stays at c forever. The same happens for receiver nodes involved in the collision event. However, since the number of receiver nodes envolved in a collision is less than those who tranmit and collide, the system randomly adds a sufficient number of idle receiver nodes and changes their state to c. By this operation, every time that there is a collision, the same number of sender and reciver nodes change their state to c and are essentially disabled forever.

Define $\tilde{M}^{(N)}(n) = (\tilde{M}_0^{(N)}(n), \tilde{M}_1^{(N)}(n), \tilde{M}_c^{(N)}(n))$ as the vector of occupancy measure for sender (or receiver) nodes where for each state $s \in \{0, 1, c\}$

$$\tilde{M}_s^{(N)}(n) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{\tilde{Y}_i^{(N)}(n)=s\}}. \tag{38}$$

Note that by definition,

$$\tilde{M}_0^{(N)}(n) + \tilde{M}_1^{(N)}(n) + \tilde{M}_c^{(N)}(n) = 1.$$

Next, we study how $\tilde{M}_s^N(n)$ evolves for $s \in \{0, 1, c\}$. Consider timeslot n, and suppose $\tilde{M}^{(N)}(n) = \tilde{M}^{(N)}$. Then, any of $\tilde{N}_0(n) = N\tilde{M}_0^{(N)}(n) = N\tilde{M}_0^{(N)}$ idle nodes (with state 0) sees $\tilde{N}_0$ idle receiving nodes (with state 0). By (37), in the next timeslot, an idle node i becomes active with total probability

$$p_0 = p_0(\tilde{M}^{(N)}) = \tilde{M}_0^{(N)}\kappa^2\beta^{(N)}, \tag{39}$$

given it is the only node that becomes active. At the same time, any of $\tilde{N}_1^{(N)}(n) = N\tilde{M}_1^{(N)}(n) = N\tilde{M}_1^{(N)}$ active nodes becomes idle independently of others with probability $\beta^{(N)}$. Considering the contributions from both idle and active nodes, we have

$$\begin{aligned} f_0^N(\tilde{M}^{(N)}) &\triangleq \mathbb{E}[\tilde{M}_0^{(N)}(n+1) - \tilde{M}_0^{(N)}(n) | \tilde{M}^{(N)}(n) = \tilde{M}^{(N)}] \\ &= -\frac{1}{N}\tilde{N}_0 p_0 + \tilde{M}_1^{(N)}\beta^{(N)} \\ &= -\Big((\tilde{M}_0^{(N)})^2\kappa^2 + (1 - \tilde{M}_0^{(N)} - \tilde{M}_c^{(N)})\Big)\beta^{(N)}. \end{aligned} \tag{40}$$

Similarly, considering the fact that only single tranmissions add to $\tilde{M}_1^{(N)}$, we obtain

$$\begin{aligned} f_1^N(\tilde{M}^{(N)}) &= \mathbb{E}[\tilde{M}_1^{(N)}(n+1) - \tilde{M}_1^{(N)}(n) | \tilde{M}^{(N)}(n) = \tilde{M}^{(N)}] \\ &= -\tilde{M}_1^{(N)}\beta^{(N)} + \frac{1}{N}\tilde{N}_0 p_0 (1-p_0)^{\tilde{N}_0 - 1} \\ &= -(1 - \tilde{M}_0^{(N)} - \tilde{M}_c^{(N)})\beta^{(N)} \\ &\quad + (\tilde{M}_0^{(N)})^2\kappa^2\beta^{(N)}(1-p_0)^{\tilde{N}_0 - 1}. \end{aligned} \tag{41}$$

Finally, we have

$$\begin{aligned} f_c^N(\tilde{M}^{(N)}) &= \mathbb{E}[\tilde{M}_c^{(N)}(n+1) - \tilde{M}_c^{(N)}(n) | \tilde{M}^{(N)}(n) = \tilde{M}^{(N)}] \\ &= \frac{1}{N}\mathbb{E}[\text{ \# of collisions} | \tilde{M}^{(N)}(n) = \tilde{M}^{(N)}] \\ &= \frac{1}{N}\big(\tilde{N}_0 p_0 - \tilde{N}_0 p_0 (1-p_0)^{\tilde{N}_0 - 1}\big) \\ &= (M_0^{(N)})^2\kappa^2\beta^{(N)}(1 - (1-p_0)^{\tilde{N}_0 - 1}). \end{aligned} \tag{42}$$

It follows from (36) that $\lim_{N\to\infty}\beta^{(N)} = 0$ and $\lim_{N\to\infty} N\beta^{(N)} = 0$, which can be used to show that

$$\begin{aligned} &\lim_{N\to\infty}(1-p_0)^{\tilde{N}_0 - 1} = \lim_{N\to\infty}(1-p_0)^{-\frac{(\tilde{N}_0 - 1)p_0}{p_0}} \\ &= \lim_{N\to\infty} e^{-(\tilde{N}_0 - 1)p_0} = \lim_{N\to\infty} e^{-(\tilde{M}_0^2\kappa^2 N\beta^{(N)} - \tilde{M}_0\kappa^2\beta^{(N)})} = 1. \end{aligned}$$

By the above limit and (40)-(42), we obtain

$$\lim_{N\to\infty}\frac{1}{\beta^{(N)}}\begin{bmatrix} f_0^N(\tilde{M}^{(N)}) \\ f_1^N(\tilde{M}^{(N)}) \\ f_c^N(\tilde{M}^{(N)}) \end{bmatrix} = F(\tilde{M}^{(N)}) \triangleq \begin{bmatrix} (\tilde{M}_0^{(N)})^2\kappa^2 + (1 - \tilde{M}_0^{(N)} - \tilde{M}_c^{(N)}) \\ -(1 - \tilde{M}_0^{(N)} - \tilde{M}_c^{(N)}) + (\tilde{M}_0^{(N)})^2\kappa^2 \\ 0 \end{bmatrix}. \tag{43}$$

It is easy to check that the system under consideration satisfies all conditions of Theorem 1 in [5]. The theorem allows us to apply mean filed results. To use the theorem, first, define the related deterministic ODE with initial vector value $\tilde{M}(0)$ as

$$\begin{aligned} \frac{d\Phi(t, \tilde{M}(0))}{dt} &= F(\Phi(t, \tilde{M}(0))) \\ \Phi(0, \tilde{M}(0)) &= \tilde{M}(0), \end{aligned} \tag{44}$$

where $F(\cdot)$ is defined in (43). In addition, define a continuous-time version of $\tilde{M}^{(N)}(t)$ denoted by $\underline{\tilde{M}}^N(t)$ by

$$\begin{aligned} &\bar{M}^N(t = n\beta^{(N)}) = \tilde{M}^{(N)}(n), \ n \in \mathbb{N} \\ &\bar{M}^N(t) \text{ is affine on } [n\beta^{(N)}, (n+1)\beta^{(N)}]. \end{aligned} \tag{45}$$

We have the following theorem:

Theorem 3. *If $\tilde{M}^{(N)}(0) \to \tilde{M}(0)$ in probability [resp. in* *mean square] as $N \to \infty$, then as $N \to \infty$*

$$\sup_{0\le t\le T} \|\tilde{\underline{M}}^{(N)}(t) - \Phi(t, \tilde{M}(0))\| \to 0$$

in probability [resp. in means square], where $\Phi(t, \tilde{M}(0))$ is the solution of (44).

This theorem suggests that for any *finite* time-horizon, we can use the deterministic ODE solution of (44) as an approximation to $\tilde{M}^{(N)}(t)$. The approximation becomes accurate in the limit. Note that instead of only one time instatnt, the theorem provides a much stronger result in terms of the sup of distances over a time interval. Before studying the ODE in (44), we need to show that the modified system closely approximates the original system. Let

$$M^{(N)}(n) = (M_0^{(N)}(n), M_1^{(N)}(n)),$$

be the vector of fraction of idle and active sender (or receiver) nodes in the original system. Let $\underline{M}^{(N)}(t)$ be the continuous-time vesrion of $M^{(N)}(n)$, defined similarly as $\tilde{\underline{M}}^{(N)}(t)$. We prove the following:

Theorem 4. *Suppose $\lim_{N\to\infty} N^2\beta^{(N)} = 0$. As $N \to \infty$,*

$$\sup_{0\le t\le T} \|\underline{M}^{(N)}(t) - \tilde{\underline{M}}^{(N)}(t)\| \to 0 \quad \textit{(in probability).}$$

Proof: We show that $\lim_{N\to\infty} N^2\beta^{(N)} = 0$ implies rare collision events even when $N \to \infty$. First, note that probability of collision at a given timeslot is given by

$$p_c^{(N)}(N_0) = 1 - (1-p_0)^{N_0} - N_0 p_0 (1-p_0)^{N_0-1}, \quad (46)$$

where N_0 is the number of idle nodes at the given timeslot, and p_0 is given by (39). It is clear that p_c^N increases with p_0 and N_0. Suppose $N_0 = N$ and $p_0 = \kappa^2\beta$ (note that $p_0 \le \kappa^2\beta$). Letting $p_c^{(N)} = p_c^{(N)}(N_0 = N)$, we have

$$\bar{p}_c^{(N)} \triangleq 1 - p_c^{(N)} = (1-\kappa^2\beta)^{N-1}(1 + (N-1)\kappa^2\beta). \quad (47)$$

Therefore, the event that there are no collisions during the first T time units, consisted of $\lfloor \frac{T}{\beta} \rfloor$, happens with probability at least

$$\begin{aligned}
\bar{p}_c^{(N)}(T) &= \bar{p}_c^{\frac{T}{\beta^{(N)}}} \\
&= \Big((1-\kappa^2\beta^{(N)})^{N-1}(1+(N-1)\kappa^2\beta^{(N)})\Big)^{\frac{T}{\beta^{(N)}}} \\
&> \Big((1-(N-1)\kappa^2\beta^{(N)})(1+(N-1)\kappa^2\beta^{(N)})\Big)^{\frac{T}{\beta^{(N)}}} \\
&= \Big(1 - ((N-1)\kappa^2\beta^{(N)})^2\Big)^{\frac{T}{\beta^{(N)}}}. \quad (48)
\end{aligned}$$

By the assumption of theorem, we have that

$$\lim_{N\to\infty} \bar{p}_c^{(N)}(T) = \lim_{N\to\infty} e^{-T(N-1)^2\kappa^2\beta^{(N)}} = 1. \quad (49)$$

The above means that when N is sufficiently large, with probability close to one, no collisions happen in a window of T time units. If this is the case, the modified system will be exactly the same as original system. This implies that

$$\begin{aligned}
&\lim_{N\to\infty} P(\sup_{0\le t\le T} \|\underline{M}^{(N)}(t) - \tilde{\underline{M}}^{(N)}(t)\| > 0) \\
&\le \lim_{N\to\infty} (1 - e^{-T(N-1)^2\kappa^2\beta}) = 0, \quad (50)
\end{aligned}$$

completing the proof. ■

B. Study of the Characteristic ODE

In order to get intution into the bahaviour of the network, we solve (44). First, note that the ODE implies that $\Phi_c(t)$ does not change with time. Therefore, by solving for $\Phi_0(t)$, we can readily obtain $\Phi_1(t)$ from $\Phi_1(t) = 1 - \Phi_0(t) - \Phi_c(t) = 1 - \Phi_0(t) - \Phi_c(0)$. We need to solve the following ODE, which is a Riccati differential equation:

$$\Phi_0' = -\kappa^2\Phi_0^2 - \Phi_0 + 1. \quad (51)$$

Letting $\Phi_p(t) = \Phi^\infty$ as one particular solution of the ODE to be defined such that

$$1 - \kappa^2(\Phi^\infty)^2 - \Phi^\infty = 0,$$

we can find the solution by assuming $\Phi(t) = \Phi^\infty + \frac{1}{\nu(t)}$:

$$\begin{aligned}
&\Phi(t) = \Phi^\infty + \\
&\Big(c(\tilde{M}(0))\; e^{(1+2\kappa^2\Phi^\infty)t} - \kappa^2(1+2\kappa^2\Phi^\infty)^{-1}\Big)^{-1}, \quad (52)
\end{aligned}$$

where $c(\tilde{M}(0))$ is a contant depending on the intial state $\tilde{M}_0(0) = \Phi(0)$ and is given by

$$c(\tilde{M}(0)) = \frac{1}{\tilde{M}_0(0) - \Phi^\infty} + \frac{\kappa^2}{1+2\kappa^2\Phi^\infty}. \quad (53)$$

The above suggests that $\Phi_0(t)$ and hence, with high probability for large N, $\underline{M}_0^{(N)}(t)$ with rate $(1+2\kappa^2\Phi^\infty)$ exponentially fast converges to Φ^∞, where Φ^∞ is the limit fraction of idle nodes. Since

$$\lim_{\kappa\to\infty} \frac{(1+2\kappa^2\Phi^\infty)}{2\kappa} = 1 \text{ and } \lim_{\kappa\to\infty} \frac{\Phi^\infty}{\kappa^{-1}} = 1. \quad (54)$$

We observe that for large κ, the convergence rate should be 2κ, and the fraction of idle nodes should be κ^{-1} at any time instant in a finite time horizon. It is clear that if at all times the fraction of idle sending nodes is $\frac{1}{\kappa}$, then distribution of idle periods becomes independent of each other and exponentially distributed with mean $\frac{1}{\kappa}$, in which case we have an ideal network where CSMA becomes memoryless. Theorem 3 and Theorem 4 suggest that this should become arbitrarily accurate for any finite amount of time as the network size increases. In the next section, we explore how we can use these results for an inifite time-horizon.

In Fig. 2 and Fig. 3, we have shown how for a 50×50 and a 400×400 network, the fraction of idle nodes changes with time. We see that as expected, for a larger network size, the simulation results better match with the analysis. Note that the curves plotted are not averaged and simply show the system behaviour over one simulation run. For these simulations, we have set $\kappa = \sqrt{5}$ and chosen $\beta^{(N)} = \frac{1}{20N\ln(N)}$, which is less strict than the requirement of Theorem 4.

C. Back to the Ideal Picture

In the previous section, we showed that one can use a simple ODE to characterize the macroscopic behaviour of the network in terms of the fraction of idle nodes. Due to the simple structure of the bipartite graphs, we can use the previous theorems to also examine the infinite time-

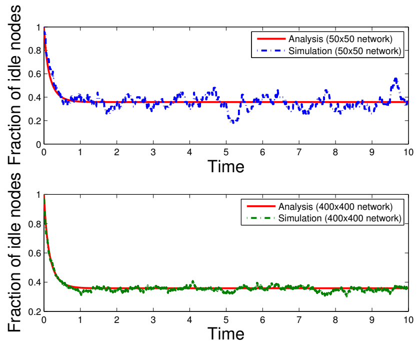

Fig. 2. Illustration of close match between simulation and analysis in terms of the fraction of idle nodes over ten packet transmission times ($\kappa = \sqrt{5}$).

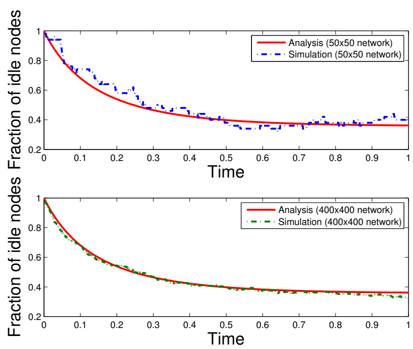

Fig. 3. Re-illustration of close match between simulation and analysis in terms of the fraction of idle nodes in the transient region ($\kappa = \sqrt{5}$).

horizon bahviour of the system. In particular, we can show the following result:

Theorem 5. *Suppose* $\lim_{N\to\infty} N^2\beta^{(N)} = 0$. *For any* $\epsilon > 0$,

$$\lim_{N\to\infty}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \mathbf{1}_{|\underline{M}_0^{(N)}(t)-\Phi^\infty|<\epsilon}\,dt = 1,\ a.s. \tag{55}$$

Proof: The proof follows by considering consequitive intervals of length T, using Theorem 3, Theorem 4, noting that the probability of collision exponentially drops to zero, taking the limit as $N \to \infty$ and then the limit as $T \to \infty$. Details are omitted due to space limitation. ∎

This theorem suggests that in the limit, the system spends all the time in states where the fraction of idle nodes is close to Φ_∞. This can be used to verify that the considered bipartite networks become ideal networks in the limit and that CSMA becomes memoryless on these networks. In Fig. 4, we have plotted the distribution of idle periods for one particular node over one simulation run as in the previous section but over 1000 time units. The analytical curve is obtained by finding Φ_∞ and assuming an exponential distribution with mean $\frac{\Phi_\infty}{1-\Phi^\infty}$. As expected, the distributions are close to exponential.

VIII. Conclusion

In this paper, we have defined ideal networks under which CSMA policies exhibit desirable probabilistic behaviour. In particular, we have shown that for these networks, the CSMA delay to access the channel, under proper scaling, becomes exponentially distributed independent of the past. With the

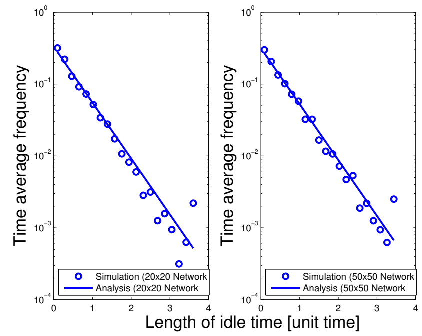

Fig. 4. Distribution of idle periods for a given node in 20×20 and 50×50 networks ($\kappa = \sqrt{5}$).

help of the mean field theory, we have shown that large symmetric bipartite graphs with small flows provide examples of the ideal networks. Although the considered example is restricting, we believe similar results should hold for large networks with sufficient symmetry properties.

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