CNFs and DNFs with exactly k Solutions

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Open problems for everyone: complexity, algorithmic, sequence generation

Warm-up: Power Sets and Counting

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Yes! For example: $S_1 = \{1, 2, 3\}$ and $S_2 = \{3, 4\}$

Then $|2^{S_1} \cup 2^{S_2}| = 8 + 4 - 2 = 10$ (they share $\emptyset$ and 3)

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Question 3: Can we find two sets S_1, S_2 such that $|2^{S_1} \cup 2^{S_2}| = 13$?

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Answer: No!

By inclusion-exclusion: $|2^{S_1} \cup 2^{S_2}| = 2^{|S_1|} + 2^{|S_2|} - 2^{|S_1 \cap S_2|}$

This is always a sum/difference of powers of 2, so it cannot equal 13.

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We Need Three Sets! For 13 elements, we need at least 3 sets.
Example: Consider $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3, 4\}$, $S_3 = \{5\}$

Question: Given a number k, what is the smallest number of sets such that the union of their power set has exactly k elements?

Antichains and Ideals

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Formal Definition: An ideal generated by family $\mathcal{S} = \{S_1, S_2, \ldots, S_\alpha\}$ is:

$$\mathrm{ID}(\mathcal{S}) = \mathrm{ID}(S_1) \cup \mathrm{ID}(S_2) \cup \cdots \cup \mathrm{ID}(S_\alpha)$$

For powerset lattice, we have $\mathrm{ID}(S_i) = 2^{S_i}$.

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For powerset lattice, we have $\text{ID}(S_i) = 2^{S_i}$.

Our Question (Reformulated):
Given k, find the size of the smallest antichain $|\mathcal{S}|$ such that $|\text{ID}(\mathcal{S})| = k$?

Connection to Boolean Satisfiability

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Consider sets: $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3, 4\}$, $S_3 = \{5\}$

Corresponding monotone DNF: $\phi = (x_4 \wedge x_5) \vee (x_1 \wedge x_5) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4)$

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Bijection:

- For term $x_4 \wedge x_5$ (from $S_1 = \{1,2,3\}$): satisfying assignments are $2^{\{1,2,3\}} = 8$
- For term $x_1 \wedge x_5$ (from $S_2 = \{2,3,4\}$): satisfying assignments are $2^{\{2,3,4\}} = 8$
- $|\text{Sol}(\phi)| = |\text{ID}(\{S_1, S_2, S_3\})|$

Our Question (Reformulated):
Given k, find the size of the smallest monotone DNF formula ϕ such that $|\text{Sol}(\phi)| = k$.

Weighted to Unweighted Model Counting

Example Setup: $\varphi = x_1 \vee x_2$ with weights $w(x_1) = \frac{5}{8}, w(x_2) = \frac{11}{16}$

Goal: Compute $\sum_{\tau \models \varphi} \prod_{x_i : \sigma(x_i)=1} w(x_i) \prod_{x_i : \sigma(x_i)=0} (1 - w(x_i))$

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Reduction Strategy:

- Add formulas with fresh variables:
  - Replace x_1 with formula $\phi_5(y_1, y_2, y_3)$ having exactly 5 solutions
  - Replace x_2 with formula $\phi_{11}(y_4, y_5, y_6, y_7)$ having exactly 11 solutions

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- Construct new formula: $\hat{\varphi} := \varphi \wedge (x_1 \leftrightarrow \phi_{10}) \wedge (x_2 \leftrightarrow \phi_{11})$
- $W(\varphi) = c \cdot |\mathrm{Sol}(\hat{\varphi}))$ for some constant c
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Key Challenge: Construct small DNFs/CNFs with exactly k solutions!

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Antichain Perspective: What is the minimum size $\alpha(k)$ of an antichain generating an ideal of size k?

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Imre Leadre (2020): "That's an interesting question. I'm not aware of any work on this problem.....Do let me know if you solve it!

Uwe Leck(2020): "Your problem looks indeed very similar to the ones we studied in our paper but, after giving it some thought, I feel that it is of quite a different nature. It's a very natural and nice question but I'm not aware of any work related to it.. How did you run into this question."

The journal version is under submission to Journal of Combinatorial Theory

Prior Work: Chain Formula Construction

[Chakraborty, Fried, M., Vardi 2015]

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Key Result: $\varphi_{k,m}$ has exactly k satisfying assignments and size $O(m)$
Gives: $\beta(k) \leq \lceil \log k \rceil = O(\log k)$ upper bound

Our Result

What is the minimum size $\alpha(k)$ of collections of sets whose power set has exactly k elements?

Main Theorem: For every $k \geq 3$,

$$\log(\mathrm{bl}(k)+1) \leq \alpha(k) \leq \min\left\{20\sqrt{\log k}\log\log k, \mathrm{bl}(k)+1\right\}$$

Observation: $\alpha(k)$ does not increase monotonically with k

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Block Binary Representation: Any $k \in \mathbb{N}$ can be written uniquely as

$$k = 1^{q_b}0^{l_b} \cdots 1^{q_2}0^{l_2}1^{q_1}0^{l_1}$$

where $q_i > 0$ and $l_j > 0$ (except possibly $l_1 = 0$)

Block Count: $\mathrm{bl}(k) = b$ (number of 1-blocks)
Example: $49 = 110001_2 = 1^2 0^3 1^1$ has $\mathrm{bl}(49) = 2$

Key Insight: $\alpha(k)$ is more closely related to $\mathrm{bl}(k)$ than to k itself!

The Block Count Connection

Key Lemma: If k can be written as $k = \sum_{i=1}^{s}(-1)^{x_i}2^{y_i}$ where $x_i \in \{0, 1\}$, then $\mathrm{bl}(k) \leq s$.

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Proof Idea: Adding/subtracting powers of 2 changes the binary representation in a controlled way:

- Adding 2^i to a number: affects at most one "block" of consecutive 1s or 0s
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- Since $t < \log(\text{bl}(k) + 1)$, we have:

$$2^t < 2^{\log(\text{bl}(k)+1)} = \text{bl}(k) + 1$$

- Therefore: $\text{bl}(k) \leq 2^t - 1 < \text{bl}(k)$ Contradiction!

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Theorem: For every $k \in \mathbb{N}$, $\alpha(k) \geq \log(\text{bl}(k) + 1)$

Proof Strategy:

- Suppose (for contradiction) that $\alpha(k) < \log(\text{bl}(k) + 1)$
- Then there exists a collection with $t = \alpha(k)$ sets whose union of power sets is exactly k.
- By inclusion-exclusion, k can be written as sum/difference of at most $2^t - 1$ powers of 2
- By our key lemma, this implies $\text{bl}(k) \leq 2^t - 1$
- Since $t < \log(\text{bl}(k) + 1)$, we have:

$$2^t < 2^{\log(\text{bl}(k)+1)} = \text{bl}(k) + 1$$

- Therefore: $\text{bl}(k) \leq 2^t - 1 < \text{bl}(k)$ Contradiction!

Insight: Numbers with many "blocks" in their binary representation are fundamentally harder to express with few terms!

Simple Upper Bound: Building Blocks

Goal: Show that $\alpha(k) \leq \mathsf{bl}(k) + 1$

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Two Fundamental Operations:
1. Splitting Lemma: $\alpha(m + k) \leq \alpha(m) + \alpha(k + 1)$

Intuition: To get an ideal of size $m + k$, combine two disjoint ideals of sizes m and $k + 1$ that only share $\emptyset$.

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Example: $\alpha(16 \cdot 13) = \alpha(208) \leq \alpha(13)$
If $\{S_1, S_2, S_3\}$ generates ideal of size 13, then $\{S_1 \cup \{a, b, c, d\}, S_2 \cup \{a, b, c, d\}, S_3 \cup \{a, b, c, d\}\}$ generates ideal of size $16 \cdot 13 = 208$.

Simple Upper Bound: The Construction

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Proof by Induction on Block Count:
Base Case: $\mathrm{bl}(k) = 1$, so $k = 1^{q_1}0^{l_1} = 2^{q_1+l_1} - 2^{l_1}$

Inductive Step: For k with $\mathrm{bl}(k) = b \geq 2$:
Write $k = 1^{q_b}0^{l_b} \cdots 1^{q_1}0^{l_1}$

$$\alpha(k) \leq \alpha(1^{q_b}0^{l_b} \cdots 1^{q_2}0^{l_2+q_1}) + \alpha(2^{q_1}) \quad \text{(Splitting)} \tag{1}$$
$$\leq (b-1) + 1 + 1 = b + 1 \quad \text{(Induction + Lifting)} \tag{2}$$

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Key Insight: Each block in the binary representation contributes roughly one generator to our construction!

Improved Upper Bound: The Challenge

Simple bound gives: $\alpha(k) \leq \mathsf{bl}(k) + 1 \in O(\log k)$

But we can do much better! Our main result: $\alpha(k) \leq O(\sqrt{\log k}\log\log k)$

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Key Observation: Most numbers have small block count!

- Numbers of form 2^q have $\mathrm{bl}(2^q) = 1$
- Numbers of form $2^a + 2^b$ have $\mathrm{bl}(2^a + 2^b) \leq 2$
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Strategy: Focus on numbers with a specific structure that are "hardest" to construct

Every k can be written as: $k = 2^{3q^2} + \gamma \cdot 2^{q^2} + \beta$
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Main Technical Result: For $m = 2^{3q^2} + \beta$ where $\beta < 2^{q^2}$:

$\alpha(m) \leq (q+1)\lceil \log q \rceil + 4q + 6 = O(q \log q)$

Open Problems for everyone!

Current Gap: Lower bound $\Omega(\log\log k)$ vs Upper bound $O(\sqrt{\log k}\log\log k)$

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Sequence Challenge: Find the sequence?

- The smallest number that can't be expressed as union of 2 power sets? 13
- The smallest number that can't be expressed as union of 3 power sets? 419
- The smallest number that can't be expressed as union of 4 power sets? ???

Questions?

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Therefore: $\alpha(k) = O(\sqrt{\log k} \log \log k)$