CSC 2410 Fall 2017, Assignment #2

Due: Tuesday Nov 7 by 10:00AM

You may consult the text. You may not consult any other materials. You may consult with each other **only on problems 1 and 2**; for each of those problems, list any students that you consulted. For the other problems, you can only consult with the instructor for this course.

Of course, each problem requires a well-written proof. Proofs that are unneccessarily lengthy might not get full marks, even if they are correct. And they might not be read thoroughly by the grader.

1. (20 pts) G is a 2-connected graph. C_1, C_2 are longest cycles in G; i.e. they each have length ℓ and there is no cycle in G of length greater than ℓ. Prove that C_1, C_2 have at least two vertices in common.

2. (25 pts)

   (a) Prove that if $|G| = n$ and $\chi(G) = t$ then $\overline{G}$ does not have a matching of size greater than $n - t$.

   (b) Use part (a) to prove that for all k, there is no k-regular graph with chromatic number k on $2k - 2$ vertices.

3. (15 pts) 5.2.13 from West.

4. (25 pts) Let H be any simple bipartite graph where each side of the bipartition has size n, and where H has maximum degree at most $t \leq \frac{n}{10}$. Show that there exists a $2t$-regular simple bipartite graph $H^{'}$ on the same vertex set, such that H is a subgraph of $H^{'}$. Describe how you could find such a graph $H^{'}$ in polytime.

   (Remark: the constant '10' is not best possible; the statement remains true for much larger upper bounds on t.)

5. (50 pts) Let G be a 4-critical graph. Recall that G has no vertices of degree less than 3. Let $L(G)$ denote the subgraph of G induced by the vertices of degree 3.

   (a) Prove that $L(G)$ cannot have an even cycle whose vertices do not form a clique. **Hint:** Colour all vertices of G except the vertices on that cycle; then argue that you can complete the colouring.

   (b) Use part (a) to prove that every block of $L(G)$ is either a clique or an odd cycle (see Def 4.1.16).

   (c) Generalize parts (a,b) for k-critical graphs, $k \geq 5$. The statements are worth part marks, but for full marks you need to prove them.

   (d) Use part (b) to prove that if G is 4-critical and not a 4-clique, then G has at least $\frac{20}{13}|V(G)|$ edges.

   Remark: Steibitz proved that $L(G)$ has at least as many components as $G \backslash L(G)$ (the graph remaining after removing $L(G)$ from G). Try to use this to prove that if G is 4-critical and not a 4-clique then G has at least $\frac{11}{7}|V(G)|$ edges. (This is not NOT part of the assignment.)