**CSC C63** Winter 2018, Assignment #4

**Due:** Wed April 4 at 11:00 PM. *You can submit it up to two days late with a penalty of 2% per day.* You must submit a .pdf file to MarkUs.

You may work with a partner to solve the problems. **You must write your own solution. You may not look at your partner's written solution. You may not consult with your partner on how to write your solution.** You can only have one partner. Neither you nor your partner can discuss this assignment with anyone else other than the course instructor and the TA. If you work with a partner then you must name your partner on your assignment.

**This assignment has 2 pages**

You can assume as fact that the following problems are NP-complete: SAT, 3-SAT, IND-SET, CLIQUE, VERTEX-COVER, SUBSET-SUM, HAMPATH (both the directed and undirected versions).

In some problems, you are asked to give a good reason for your answer. Your mark will be based on the strength of the reason. A proof is a very strong reason, but is not always possible. Proving something like "otherwise, P=NP" is not quite as strong as a proof, but is still a very strong reason. Saying "I think it's NP-complete because it looks a bit like CLIQUE" is a weak reason.

The graphs in this assignment are all undirected.

1. **(30 pts)** For each of the following, state whether you think that the problem is NP-complete. Justify your answer by either proving that it is NP-complete, or giving a good reason to think that it is not.

   (a) **LONG-PATH**
   **Input:** A graph $G$ with two specified vertices $u, v$, and a target $T$.
   **Question:** Does $G$ have a path from $u$ to $v$ of length at least $T$?
   Clarification: A path does not repeat any vertices. Its length is the number of edges.

   (b) **10-VERTEX-COVER**
   **Input:** A graph $G$.
   **Question:** Does $G$ have a vertex cover of size at most 10?

   (c) **CLIQUE-AND-IND-SET**
   **Input:** A graph $G$ and a target $T$.
   **Question:** Does $G$ have a clique and an independent set whose sizes sum to at least $T$?
   *Clarification:* The clique and independent set do not have to be disjoint. If a vertex is in both, then it contributes two to the sum of the sizes.

2. **(20 pts)** Exercise 7.29 of Sipser (3rd edition). This is Problem 7.27 of Sipser (2nd edition). Here, you will show that 3COLOR is NP-complete by reducing it from 3-SAT. You can skip the step where you show that 3COLOR$\in$NP.

   *Comment:* Moore-Mertens provides a different proof that 3COLOUR is NP-complete by reducing it from NAE-SAT. It may be helpful to read that, but for your assignment you need to give a reduction from 3-SAT using the subgraphs provided in the Sipser exercise.

3. **(15 pts)**

   A subset $A$ of the vertices of a graph is a *maximal clique* if (i) $A$ is a clique and (ii) $A$ is not a subset of a bigger clique.

   Show that the following problem can be solved in PSPACE:

   **MANY-MAXIMAL-CLIQUES**
   **Input:** A graph $G$ and an integer $T$.
   **Question:** Does $G$ contain at least $T$ maximal cliques?

4. **(25 pts)** In this problem, you need to verify the answer to a multiplication.

   **MULTIPLICATION**
   **Input:** Three integers $a, b, c$ presented in binary
   **Question:** Is $a \times b = c$?

   Prove that MULTIPLICATION is in $L$.

   *Clarification:* $a, b, c$ are written in binary on the input tape. They are separated by a hash-character. Note that the size of the input is the total number of digits in $a, b, c$. So for your algorithm to run in $L$, the space used must be a constant multiple of *the logarithm of the total number of digits*.

5. **(20 pts)** A *walk from $s$ to $t$ of length $\ell$* in a graph is a sequence of vertices $v_0, ..., v_\ell$ where $v_0 = s, v_\ell = t$ and $v_i, v_{i+1}$ is joined by an edge for every $0 \leq i \leq \ell - 1$. So the only difference between a walk and a path is that a walk may repeat vertices.

   **WALKS-OF-MANY-LENGTHS**
   **Input:** An undirected graph $G$ with two specified vertices $s, t$, and two integers $1 \leq L_1 \leq L_2 \leq n$, where $n$ is the number of vertices in $G$.
   **Question:** Does $G$ have a walk from $s$ to $t$ of length $\ell$ for every $L_1 \leq \ell \leq L_2$?

   So, for example, if $L_1 = 6, L_2 = 9$ then we are asking whether there are four walks of lengths 6,7,8 and 9.

   Prove that WALKS-OF-MANY-LENGTHS $\in$ NL by providing a logspace NDTM that accepts WALKS-OF-MANY-LENGTHS.

   **Note:** Since $L_1 \leq L_2 \leq n$, we know that the input file has size at least $L_2$. So in this problem the NDTM is allowed to use $O(\log L_2)$ bits of space. (Do you understand why?)