CSC C63 Final Exam

April 10, 2015

NAME:

Calculators are not permitted (nor would they be useful).

This is a closed book exam.

#1	/24	#5	/12
#2	/16	#6	/12
#3	/24	#7	/15
#4	/35		
			/138

13 pages

You can assume that the following problems are NP-complete.

CNF-SAT
Input: A CNF boolean formula, F.
Question: Is F satisfiable?

3-SAT
Input: A CNF boolean formula, F, in which every clause has exactly 3 literals.
Question: Is F satisfiable?

IND-SET
Input: A graph G, and an integer K.
Question: Does G have an independent set of size K?

CLIQUE
Input: A graph G, and an integer K.
Question: Does G have a clique of size K?

VERTEX-COVER
Input: A graph G, and an integer K.
Question: Does G have a vertex cover of size K?

3-COLOUR
Input: A graph G.
Question: Does G have a proper 3-colouring?

SUBSET-SUM
Input: A list of positive integers $x_1, ..., x_n$, and a target T.
Question: Is there a subset of the integers which sums to T?

HAM-PATH
Input: A graph G, with two specified vertices u, v.
Question: Does G have a Hamilton path from u to v?

HAM-CYCLE
Input: A graph G.
Question: Does G have a Hamilton cycle?

KNAPSACK
Input: A list of weights $w_1, ..., w_n$ and values $v_1, ..., v_n$ along with a capacity W and a target T. All numbers are positive integers.
Question: Is there a subset of indices $I \subset \{1, ..., n\}$ such that $\sum_{i \in I} w_i \leq W$ and $\sum_{i \in I} v_i \geq T$?

1. **(24 pts)** For each of the following statements, say that it is one of:

   - True.
   - False.
   - No one knows, but most experts think it is True.
   - No one knows, but most experts think it is False.

   Do not explain your answer.

   (a) P = NP

   (b) PSPACE = NPSPACE

   (c) P=EXPTIME

   (d) SUBSET-SUM $\in$ co-NP

   (e) FACTOR $\in$ P

   (f) PATH $\in$ L

   (g) The Post Correspondence Problem is in EXPTIME.

   (h) There is an algorithm to determine whether an algorithm will sort a list of numbers.

2. **(16 pts)** Only short answers are required here.

   (a) **(4 pts)** State Church's Thesis

   (b) **(4 pts)** Describe the tapes used by a Turing Machine that provides a logspace reduction; i.e. that demonstrates $A \leq_L B$.

(c) **(4 pts)** Give an example of a problem that you know is impossible to solve with an algorithm.

(d) **(4 pts)**

The following algorithm tests Goldbach's Conjecture for specific numbers. It takes an even number $2x$ and checks whether it is the sum of two primes.

Input x (a positive integer)
For $p = 1$ to x
    Check whether p is a prime using the polytime prime-testing algorithm
    Check whether $2x - p$ is a prime using the polytime prime-testing algorithm
    If p and $2x - p$ are both prime then Accept
Reject

Is this a polytime algorithm? Explain.

3. **(24 pts)** Consider the set

$$A = \{(< P >, k) |\text{ there is no } x \geq 2k \text{ such that } P(x) \text{ halts and returns } x + k + 1.\}$$

P denotes a Turing Machine and k is a **positive** integer.

(a) **(15 pts)** Is A decidable? Prove your answer.

(b) **(3 pts)** State $\overline{A}$ in the form $\overline{A} = \{...\}$.

(c) **(3pts)** Is $\overline{A}$ decidable? Prove your answer.

(d) **(3pts)** Either A or $\overline{A}$ is recognizable. Which one? You do not have to prove your answer.

4. **(35 pts)** Each of the following problems is in NP; you do not have to prove this (until later in the exam).

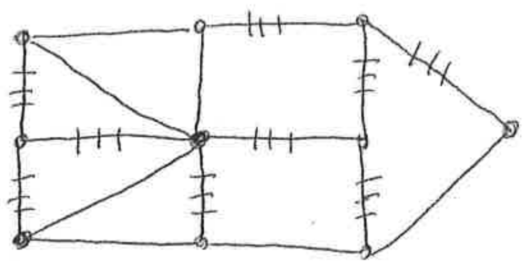
   For each problem, either prove that it is NP-complete or prove that it is in P.

   (a) **(10 pts)**
   HUGE-IND-SET
   Input: A graph G.
   Question: Does G have an independent set of size $n-1$?

   (b) **(10 pts)**
   EVEN-SUBSET-SUM
   Input: A list of positive integers $x_1, ..., x_n$, and a target T which is an **even number**.
   Question: Is there a subset of the integers which sums to T?

(c) **(15 pts)**

A (1,3)-tree is a tree in which every vertex has degree either 1 or 3. A tree T is a spanning (1,3)-tree of a graph G if (i) T is a (1,3)-tree ; (ii) T is a subgraph of G; and (iii) T contains every vertex of G. For example, the marked edges indicate a spanning (1,3)-tree in the following graph:



SPANNING (1,3)-TREE
Input: A graph G.
Question: Does G have a spanning (1,3)-tree?

Hint: This one is NP-complete. Consider a (1,3)-tree where every degree 3 vertex has at least one degree 1 neighbour.

5. **(12 pts)** Choose any two problems from the previous question that you think are NP-complete and prove that they are both in NP. For one problem, do this using a verification algorithm. For the other problem, do this using a nondeterministic Turing Machine.

6. **(12 pts)** A CNF formula F is *almost satisfiable* if (i) it is not satisfiable and (ii) every clause has the property that if it is removed then the resulting formula is satisfiable. For example, the following formula is almost satisfiable:

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) \wedge (\overline{x_2})$$

If we remove the first clause, then $x_1 = F, x_2 = F$ is a solution; If we remove the second clause, then $x_1 = T, x_2 = F$ is a solution; if we remove the third clause, then $x_1 = T, x_2 = T$ is a solution.

ALMOST-SAT
Input: A CNF formula F.
Question: Is F almost satisfiable?

Show that ALMOST-SAT is in PSPACE.

7. **(15 pts)**

(a) **(10pts)** A is a recognizable language. A^{++} is the set of strings formed by concatenating three strings from A; i.e.

$$A^{++} = \{a_1a_2a_3 : a_1, a_2, a_3 \in A\}$$

Prove that A^{++} is recognizable.

(b) **(5pts)** Is the following statement true? Prove your answer.

If A is recognizable and $A \cup B$ is recognizable then B is recognizable.