$(\Delta - k)$-critical graphs[1]

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Abstract

Every graph G of maximum degree Δ is $(\Delta+1)$-colourable and a classical theorem of Brooks states that G is not Δ-colourable iff G has a $(\Delta+1)$-clique or $\Delta = 2$ and G has an odd cycle. Reed extended Brooks' Theorem by showing that if $\Delta(G) \geq 10^{14}$ then G is not $(\Delta-1)$-colourable iff G contains a Δ-clique. We extend Reed's characterization of $(\Delta-1)$-colourable graphs and characterize $(\Delta-2)$, $(\Delta-3)$, $(\Delta-4)$ and $(\Delta-5)$-colourable graphs, for sufficiently large Δ, and prove a general structure for graphs with χ close to Δ. We give a linear time algorithm to check the $(\Delta-k)$-colourability of a graph, for sufficiently large Δ and any constant k.

Key words: Critical Graphs, Graph Colouring, Probabilistic Methods, Chromatic Number, Maximum Degree

1 Introduction

Throughout the paper, by the word graph we mean a simple undirected graph. $\delta(G)$, $\Delta(G)$ and $\chi(G)$ denote the minimum degree, the maximum degree and the chromatic number of a graph G. A graph G is called critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such a G is c-critical if $\chi(G) = c$.

It is easy to show that every graph G is $(\Delta + 1)$-colourable. In 1941, Brooks characterized those graphs with $\chi = \Delta + 1$.

Theorem 1.1 (Brooks) *For every graph G, $\chi(G) \leq \Delta + 1$. Moreover, $\chi(G) = \Delta + 1$ iff $\Delta(G) \neq 2$ and G contains a $(\Delta + 1)$-clique, or $\Delta(G) = 2$ and G contains an odd cycle.*

It is easy to see that, in fact, G has a $(\Delta+1)$-clique iff it has a $(\Delta+1)$-clique as a connected component. The same is true when $\Delta = 2$ for odd cycles. Thus, Brooks' characterization of graphs with $\chi = \Delta + 1$ yields an efficient algorithm to check Δ-colourability of a graph. In 1998, Reed [13] proved that

Theorem 1.2 *If G is a graph of maximum degree $\Delta \geq 10^{14}$, then $\chi(G) \geq \Delta$ iff G contains a Δ-clique as a subgraph.*

In this paper, we will extend Brooks' and Reed's characterizations of graphs with $\chi \geq \Delta + 1$ and $\chi \geq \Delta$ to graphs with $\chi \geq \Delta - 1$, $\chi \geq \Delta - 2$, $\chi \geq \Delta - 3$ and $\chi \geq \Delta - 4$, for sufficiently large Δ. We will show that, for large Δ, the number of non-isomorphic $(\Delta - 1)$-critical, $(\Delta - 2)$-critical, $(\Delta - 3)$-critical and $(\Delta - 4)$-critical graphs are 2, 4, 26 and 420, respectively. In doing so, we provide a general structure for graphs with chromatic number close to their maximum degree. Also, we will give a lower bound of 17036 on the number of $(\Delta - 5)$-critical graphs.

We will use these, and similar, characterizations to provide a linear time algorithm to check $(\Delta - k)$-colourability of a graph, for any constant k and sufficiently large Δ. We will also show how to apply our main results in Section 3 to solve the asymptotic version of Problem 4.7 from Jensen and Toft [6].

Our main tool is the following theorem by Molloy and Reed [9]:

Theorem 1.3 *There exists $\epsilon > 0$ such that if G is a graph of maximum degree Δ and $k \leq \epsilon\sqrt{\Delta}$ then $\chi(G) > \Delta - k$ iff there exists a $(\Delta - k + 1)$-critical subgraph H of G with at most $\Delta + \sqrt{\Delta}$ vertices.*

They do not specify ϵ but it is very small, less than 10^{-6}.

This theorem implies that critical graphs with sufficiently large Δ and chromatic number close to Δ have bounded size. This will be of great help to us, as it converts the study of such graphs to the study of critical graphs with relatively small size and high chromatic number.

It is worth noting that Molloy and Reed [11] later improved their result, showing that

Theorem 1.4 *There exists Δ_0 such that for all $\Delta \geq \Delta_0$ and $k^2 + 2k < \Delta$, if G has maximum degree Δ and $\chi(G) > \Delta - k$, then there exists a subgraph H of G such that $|H| \leq \Delta + 1$ and $\chi(H) > \Delta - k$.*

This bound is tight; i.e. they showed:

Theorem 1.5 *For Δ arbitrarily large and for any k with $k^2+2k \geq \Delta$, there exist graphs G with maximum degree Δ and with $\chi(G) > \Delta - k$ for which every subgraph of size at most $\Delta + 1$ is $(\Delta - k)$-colourable. Furthermore, for Δ arbitrarily large and for any k with $k^2 + 2k > \Delta$, there are arbitrarily large $(\Delta - k)$-critical graphs G.*

2 Preliminaries

The following standard facts will frequently be used in our proofs.

Fact 2.1 *Every c-chromatic graph contains a c-critical subgraph.*

Fact 2.2 *If G is a c-critical graph then $\delta(G) \geq c - 1$.*

The join of disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined to be the graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in V_2\}$. The join of a graph $G_1 = (V_1, E_1)$ and a set of vertices $V_2 = \{v_1, \ldots, v_t\}$ is defined as the join of G_1 and $G_2 = (V_2, \emptyset)$. A vertex v is said to be joined to the set of vertices $V = \{v_1, \ldots, v_t\}$ if $V \subseteq N(v)$. A vertex v in a graph G is called a dominating vertex if it is joined to all other vertices in G, i.e. if $N(v) = V(G) - \{v\}$.

Fact 2.3 *If G is the join of two disjoint graphs G_1 and G_2 then $\chi(G) = \chi(G_1) + \chi(G_2)$. Moreover, G is critical iff G_1 and G_2 are critical.*

The next lemma follows immediately from a theorem of Gallai[4]. (For an alternative reference in English, it also follows from Lemma 2 of [8].)

Lemma 2.4 *For any critical graph G with $\chi(G) > \frac{|G|}{2}$, G is the join of graphs $H_1, H_2, \ldots, H_t$, where*

(a) $\chi(H_i) \leq \lceil \frac{|H_i|}{2} \rceil$, $1 \leq i \leq t$;

(b) $\sum_{i=1}^{t} \chi(H_i) = \chi(G)$.

This yields the following lemma, which shows that the graphs which interest us must be cliques joined to very small critical subgraphs.

Lemma 2.5 *Consider a graph G of maximum degree Δ with $\chi(G) = \Delta - k > \frac{3}{5}|G|$. If H is a $(\Delta - k)$-critical subgraph of G, then either H is a $(\Delta - k)$-clique or H is a non-empty clique Q joined to a critical graph $H^{'}$ with no dominating vertices, where $5 \leq |H^{'}| \leq \frac{5}{2}(k+1)$ and $\chi(H') \leq \frac{3}{5}|H^{'}|$. Equality holds iff $H^{'}$ is the join of $\frac{k+1}{2}$ disjoint 5-cycles.*

Proof. Since $\chi(H) > \lceil \frac{|H|}{2} \rceil$ and H is critical, Lemma 2.4 implies that H is the join of graphs $H_1, H_2, \ldots, H_t$, where $t > 0$. Note that since H is critical, each H_i is $\chi(H_i)$-critical. Trivially, there are no 1-critical graphs on at least 2 vertices, nor 2-critical graphs on at least 3 vertices. Thus, if $|H_i| \neq 1$, then $\chi(H_i) \geq 3$ and so $|H_i| \geq 5$, and we have $\chi(H_i) \leq \lceil \frac{|H_i|}{2} \rceil \leq \frac{3}{5}|H_i|$. Thus,

$$\chi(H) = \sum_{i=1}^{t} \chi(H_i) \leq |\{H_i : |H_i| = 1\}| + \sum_{|H_i| \geq 5} \frac{3}{5}|H_i|.$$

Note that each H_i of size 1 corresponds to a dominating vertex in H. Also, the dominating vertices in H form a clique, Q, which is joined to $H^{'}$, the rest of H. So, $|H'| = \sum_{|H_i| \geq 5} |H_i|$, $\chi(H) = |Q| + \chi(H')$, and $\chi(H') \leq \frac{3}{5}|H'|$.

Furthermore, since $\chi(H) = \chi(G) > \frac{3}{5}|G| \geq \frac{3}{5}|H|$, there is at least one vertex in Q. That vertex has degree at most Δ and so $|Q| + |H^{'}| \leq \Delta + 1$. Since $|Q| + \frac{3}{5}|H^{'}| = \chi(H) = \Delta - k$, this yields $\frac{2}{5}|H^{'}| \leq k + 1$, i.e. $|H^{'}| \leq \frac{5}{2}(k+1)$.

Finally, it is straightforward to verify that this proof yields $|H^{'}| < \frac{5}{2}(k+1)$ and $\chi(H^{'}) < \frac{3}{5}|H^{'}|$ unless each H_i is either a single vertex or a 5-cycle, in which case $H^{'}$ is the join of $\frac{k+1}{2}$ disjoint 5-cycles. ■

3 Graphs with chromatic number at least $\Delta - k$, $1 \leq k \leq 5$

In this section, we will obtain characterizations of graphs with chromatic number at least $\Delta - 1$, at least $\Delta - 2$, at least $\Delta - 3$ and at least $\Delta - 4$. We also discuss graphs with chromatic number at least $\Delta - 5$.

Theorem 3.1 *There exists Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$, we have: $\chi(G) \geq \Delta - 1$ iff G contains a subgraph isomorphic to either*

(i) a $(\Delta - 1)$-clique or

(ii) a $(\Delta - 4)$-clique joined to a C_5 (a cycle on 5 vertices).

Proof. We will take $\Delta_0 \geq 10^{14}$, so if $\chi(G) > \Delta - 1$ then by Theorem 1.2, G contains a $(\Delta - 1)$-clique. So, assume that $\chi(G) = \Delta - 1$ and that G does not contain a $(\Delta - 1)$-clique. We will show that G contains a $(\Delta - 4)$-clique joined to a C_5.

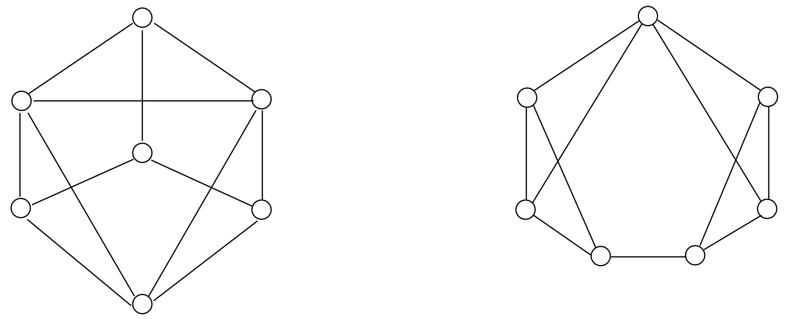

Figure 1: The only two critical graphs on 7 vertices with $\chi = 4$ with no dominating vertex.

We also take $\Delta_0 \geq 4\epsilon^2$ where ϵ is as in Theorem 1.3. Thus, applying Theorem 1.3 with $k = 2$, we can consider a $(\Delta - 1)$-critical subgraph H of G which has size at most $\Delta + \sqrt{\Delta}$. Since $\chi(H) = \Delta - 1 > \frac{3}{5}|H|$, Lemma 2.5 implies that H is a clique Q joined to a critical graph $H^{'}$ where $|H^{'}| = 5$ and $\chi(H^{'}) \leq 3$. The only possibility is that $H^{'}$ is a 5-cycle, which completes the proof. ■

Theorem 3.2 *There exists Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$, we have: $\chi(G) \geq \Delta - 2$ iff it contains a subgraph isomorphic to*

(i) a $(\Delta - 2)$-clique or

(ii) a $(\Delta - 5)$-clique joined to a C_5 or

(iii) a $(\Delta - 6)$-clique joined to one of the two graphs shown in figure 1.

Proof. Applying Theorem 1.3 with $k = 3$, and taking $\Delta_0 \geq 9\epsilon^2$, G has a $(\Delta - 2)$-critical subgraph H of size at most $\Delta + \sqrt{\Delta}$. Assume that H is of maximum degree Δ. Otherwise, we can apply Theorem 3.1 on H where

$\chi(H) \geq \Delta(H) - 1$ and show that H contains either a $(\Delta - 2)$-clique or a $(\Delta - 5)$-clique joined to a C_5 (by taking Δ_0 to be at least as high as the Δ_0 from Theorem 3.1).

Again applying Lemma 2.5, we see that either H is a $(\Delta - 2)$-clique or H is a clique joined to a critical graph H' with no dominating vertex and of size between 5 and 7 and with chromatic number at most 4. Furthermore, if $|H'| < 7$ then H' has chromatic number at most 3. H' has no dominating vertices, so the only possibilities are that H' is a 5-cycle or a 4-critical graph on 7 vertices. The former possibility implies that H is a $(\Delta-5)$-clique joined to a C_5. For the latter possibility, we refer to Toft [15] who proved that there are only two 4-critical graphs on 7 vertices (shown in figure 1); neither of them contains a dominating vertex. ∎

Theorem 3.3 *There exists Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$, we have: $\chi(G) \geq \Delta - 3$ iff G contains a subgraph isomorphic to one of the following 26 graphs:*

i) a $(\Delta - 3)$-clique;

ii) a $(\Delta - 6)$-clique joined to a C_5;

iii) a $(\Delta - 7)$-clique joined to one of the graphs shown in figure 1;

iv) a $(\Delta - 6)$-clique joined to a C_7;

v) a $(\Delta - 7)$-clique joined to one of the graphs shown in figure 2;

vi) a $(\Delta - 8)$-clique joined to one of the graphs shown in figure 3;

vii) a $(\Delta - 9)$-clique joined to the graph shown in figure 4.

Proof. Applying Theorem 1.3 with $k = 4$, and taking $\Delta_0 \geq 16\epsilon^2$, G has a $(\Delta - 3)$-critical subgraph H of size at most $\Delta + \sqrt{\Delta}$. We can assume that H is of maximum degree Δ as otherwise, we can apply Theorem 3.2 on H as $\chi(H) \geq \Delta(H) - 2$ (by taking Δ_0 to be at least as high as the Δ_0 from Theorem 3.2). So assume that H is of maximum degree Δ and is not $(\Delta - 3)$-clique. We will show that H contains a subgraph isomorphic to one of the graphs stated in iv), v), vi) and vii).

By Lemma 2.5, H is a clique Q joined to a graph H' where $|H'| \leq 10$, and H' has no dominating vertex. Since H' is critical, $\chi(H') \geq 3$. Since every vertex of Q is dominating, and H has a vertex of degree Δ, that vertex must be in Q. This yields $|H| = |Q| + |H^{'}| = \Delta + 1$. Also, $\chi(H) = |Q| + \chi(H^{'}) = \Delta - 3$. Subtracting these equations yields $\chi(H') = |H'| - 4$. Thus, we may have one of the following cases:

Case 1: $\chi(H') = 3$. This implies that H' is a 3-critical graph on $\chi(H') + 4 = 7$ vertices. Therefore, H' is a 7-cycle.

Case 2: $\chi(H') = 4$. This implies that $|H'| = \chi(H') + 4 = 8$. Toft proved that there are 5 non-isomorphic 4-critical graphs on 8 vertices [15]. One of the five contains a dominating vertex. The others are shown in figure 2.

Case 3: $\chi(H') = 5$. This implies that $|H'| = \chi(H') + 4 = 9$. Jensen and Royle showed that there are 21 non-isomorphic 5-critical graphs on 9 vertices [5]. Five of them contain a dominating vertex. The others are shown in figure 3.

Case 4: $\chi(H') = 6$. This implies that $|H'| = \chi(H') + 4 = 10$. Such an $|H^{'}|$ has parameters which are tight for Lemma 2.5 and so $H^{'}$ must be the join of two 5-cycles (see figure 4). ■

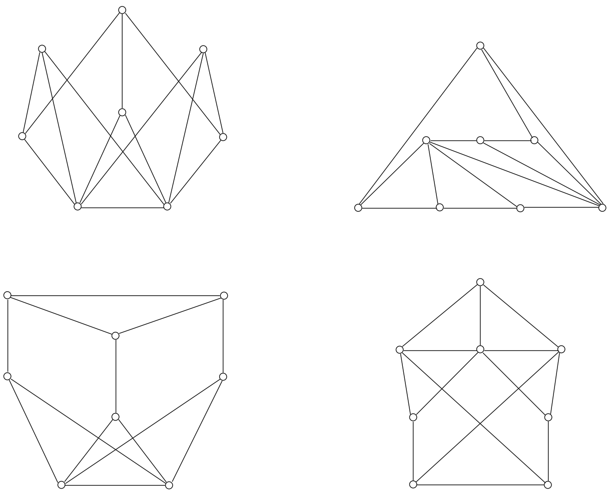

Figure 2: 4-critical graphs on 8 vertices with no dominating vertex.

Theorem 3.4 *There exists Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$, we have: $\chi(G) \geq \Delta - 4$ iff G contains a subgraph isomorphic to one of 420 non-isomorphic $(\Delta - 4)$-critical graphs.*

The proof of Theorem 3.4 relies heavily on a list computed by Royle [14] of all small critical graphs.

Proof. Applying Theorem 1.3 with $k = 5$, and taking $\Delta_0 \geq 25\epsilon^2$, G has a $(\Delta - 3)$-critical subgraph H of size at most $\Delta + \sqrt{\Delta}$. We can assume that H is of maximum degree Δ as otherwise, we can apply Theorem 3.3 on H as $\chi(H) \geq \Delta(H) - 2$ (by taking Δ_0 to be at least as high as the Δ_0 from Theorem 3.3). So assume that H is of maximum degree Δ and is not a $(\Delta - 4)$-clique.

By Lemma 2.5, H is a clique Q joined to a graph H' where $|H'| \leq 12$, and H' has no dominating vertex. Since H' is critical, $\chi(H') \geq 3$. Each vertex of Q is dominating, and H has a vertex of degree Δ, so that vertex must be in

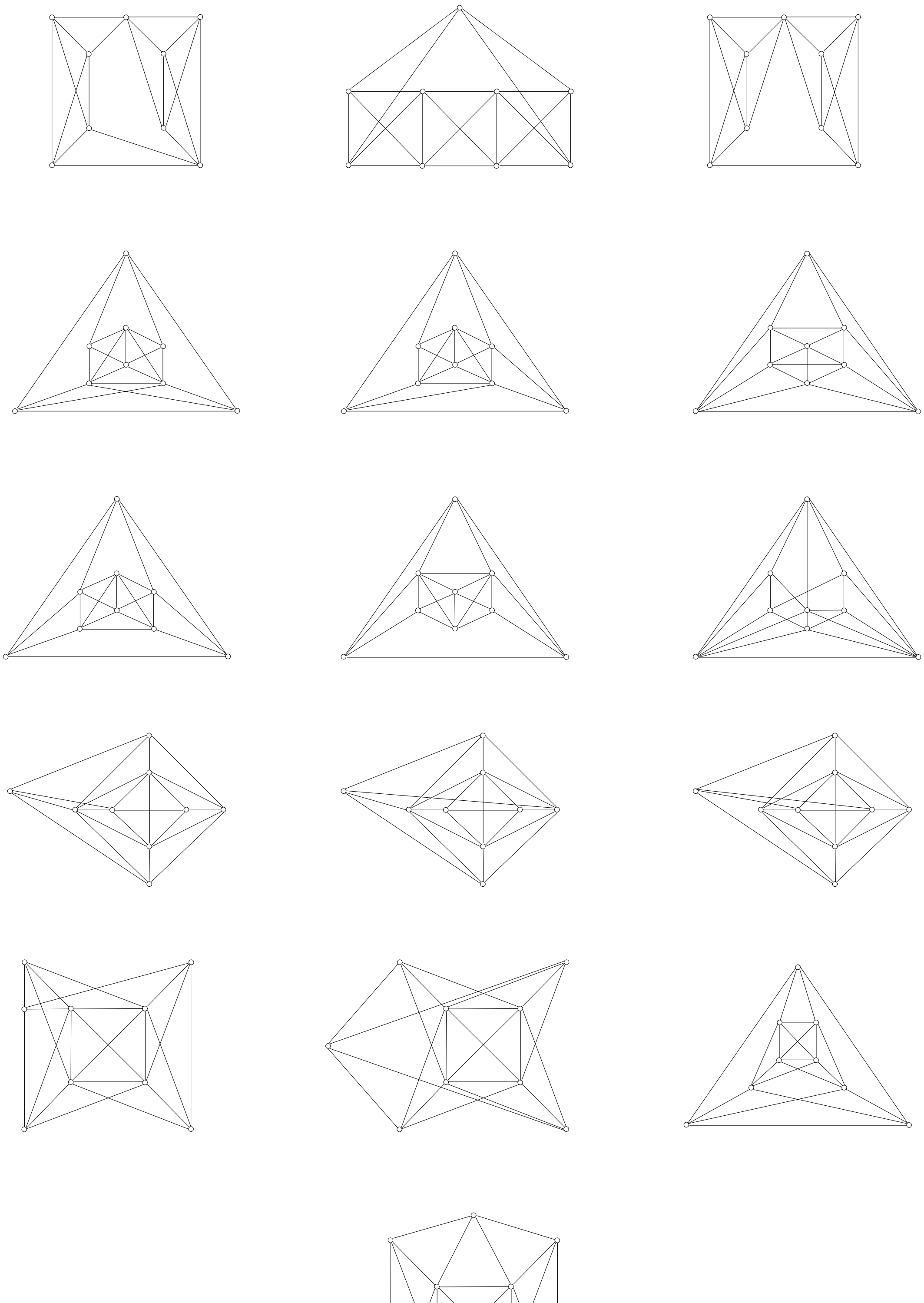

Figure 3: 5-critical graphs on 9 vertices with no dominating vertex.

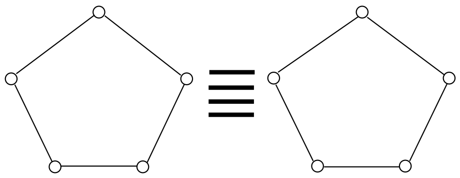

Figure 4: The only 6-critical graph on 10 vertices with no dominating vertex.

Q. This yields $|H| = |Q| + |H^{'}| = \Delta + 1$. Also, $\chi(H) = |Q| + \chi(H^{'}) = \Delta - 4$. Subtracting these equations yields $\chi(H') = |H'| - 5$. Thus, we may have one of the following cases:

i) a $(\Delta - 8)$-clique joined to a 4-critical graph on 9 vertices with no dominating vertex. There are 21 non-isomorphic 4-critical graphs on 9 vertices[14] and none of them has a dominating vertex;

ii) a $(\Delta - 9)$-clique joined to a 5-critical graph on 10 vertices with no dominating vertex. There are 162 non-isomorphic 5-critical graphs on 10 vertices[14] and 141 of them have no dominating vertices;

iii) a $(\Delta - 10)$-clique joined to a 6-critical graph on 11 vertices with no dominating vertex. There are 393 non-isomorphic 6-critical graphs on 11 vertices[14] and 230 of them have no dominating vertices;

iv) a $(\Delta - 11)$-clique joined to a 7-critical graph on 12 vertices with no dominating vertex. There are 395 non-isomorphic 7-critical graphs on 12 vertices[14] and only 2 of them have no dominating vertices;

Considering all the above cases, when Δ is sufficiently large, there are 420 non-isomorphic $(\Delta - 4)$-critical graphs. ∎

To obtain analogous results for graphs with chromatic number at least $\Delta - 5$ requires enumerating all possibilities for $H^{'}$ where $H^{'}$ is a critical graph of size at most 15 with $\chi(H) = |H^{'}| - 6$.

The list computed by Royle [14] of all small critical graphs shows that there are 17036 non-isomorphic 6-critical graphs on 12 vertices, each of which can be joined to a $(\Delta - 11)$-clique to form a $(\Delta - 5)$-critical graph. The number of 7-critical graphs on 13 vertices, 8-critical graphs on 14 vertices and 9-critical graphs on 15 vertices is not known. Thus, we only have a lower bound on the number of $(\Delta - 5)$-critical graphs.

Chromatic number	number of critical graphs
Δ	1
$\Delta - 1$	2
$\Delta - 2$	4
$\Delta - 3$	26
$\Delta - 4$	420
$\Delta - 5$	> 17000

Table 1: Number of $(\Delta - k)$-critical graphs for $k \leq 5$ and Δ sufficiently large.

4 A linear time algorithm for $(\Delta - k)$-colourability

In general, determining whether a graph G of maximum degree Δ is c-colourable is NP-hard though for some specific pairs of c and Δ it is easy. We know that every graph G is $(\Delta + 1)$-colourable. It is also easy to determine whether G is Δ-colourable: using Brooks' Theorem, we only need to search for a component which is a $(\Delta + 1)$-clique and, if $\Delta = 2$, also check for components which are odd cycles. The main result of this section is:

Theorem 4.1 *For any constant k, there is a linear time algorithm which will test $(\Delta(G) - k)$-colourability of any input graph G with $\Delta(G) \geq \Delta_0 = \Delta_0(k)$.*

In [9], Molloy and Reed provided a linear time algorithm to recognize k-colourability for the range of k, Δ from Theorem 1.3, so long as $\Delta = O(1)$. For general Δ, they proved that there is a polytime algorithm so long as $k \leq O(\log \Delta)$, and noted in [11] that this is best possible unless there is a subexponential time algorithm for general graph colourability. In [11], they extended this to the range of Theorem 1.4; in fact they allow k to be slightly larger than in Theorem 1.4:

Theorem 4.2 *For any constant $\Delta \geq \Delta_0$ and k where $k^2 + k \leq \Delta$, $(\Delta - k)$-colourability of graphs with maximum degree Δ can be determined in linear time.*

This is sharp, in the sense that Emden-Weinert, Hougardy and Kreuter [2] proved that for all values of Δ the problem is NP-complete when $k^2 + k > \Delta$ (unless, of course, $k = \Delta - 2$).

Theorem 4.1 extends Theorem 4.2 to prove that $(\Delta - k)$-colourability of G can be determined in linear time, even if $\Delta \neq O(1)$, so long as $k = O(1)$ and Δ is sufficiently large.

The key step in these algorithms is a decomposition of the graph into dense sets. More formally, it is shown that the vertex set of every graph can be partitioned into disjoint sets $D_1, \cdots, D_t, S$, such that each D_i is dense, Δ-clique-like and every vertex in S has a sparse neighborhood. First, we give formal definitions for such a decomposition. A vertex v in a graph G of maximum degree Δ is d-dense if there are at least $\binom{\Delta}{2} - d\Delta$ edges in the subgraph induced by its neighborhood. Otherwise, v is d-sparse. Note that if d is sufficiently small and v is d-dense, this implies that v has nearly Δ neighbors and very few missing edges in its neighborhood.

Definition 1 *A d-dense decomposition of a graph G of maximum degree Δ is a partition of $V(G)$ into disjoint sets $D_1, \ldots, D_t, S$, such that:*

(a) a vertex v is in D_i if and only if it has at least $\frac{3\Delta}{4}$ neighbors in D_i;

(b) there are at most $8d\Delta$ edges between each D_i and the rest of the graph;

(c) the number of vertices in each D_i is between $\Delta+1-8d$ and $\Delta+1+4d$;

(d) if a vertex v is in S then it is d-sparse.

We are interested in such decompositions since it has been proved that if we can colour each dense set D_i with $\Delta - k$ colours, where k is sufficiently small, a $(\Delta - k)$-colouring can also be found for the whole graph (see Theorem

4.4). The following theorem [12] shows that for appropriate values of d, there exists a d-dense decomposition for all graphs.

Theorem 4.3 *A graph G of maximum degree Δ has a d-dense decomposition, for all $d \leq \frac{\Delta}{100}$. Moreover, such a decomposition can be found in linear time.*

For a proof and description of the algorithm, see [10] or [12]. Let G be a graph with $\chi(G) > (\Delta - k)$, where $k^2 - k \leq \Delta$. A key step in the proof of Theorem 1.3 was:

Theorem 4.4 *For any d-dense decomposition of G with $d = 1000\sqrt{\Delta}$, the subgraph H' from Theorem 1.3 is contained in a dense set D_i.*

This implies Theorem 4.2 as follows: consider a d-dense decomposition of G, where $d = 1000\sqrt{\Delta}$. We can find such a decomposition in linear time using the Theorem 4.3. By Theorem 4.4, it would be sufficient to prove that $(\Delta - k)$-colourability of a d-dense set D can be determined in constant time. From the above definition, $|D| \leq \Delta + 1 + 4d$. Since both Δ and k are constants, $(\Delta - k)$-colourability of the graph induced on D can be checked in constant time, using exhaustive search. Thus, $(\Delta - k)$-colourability of G can be determined in linear time.

The first step in our proof is to show that our dense sets are extremely close to being cliques.

Lemma 4.5 *Consider a d-dense set D, where $d \leq \frac{\Delta}{100}$ and Δ is sufficiently large. If $\chi(D) > \Delta - k \geq \frac{3}{5}\Delta$ then*

(a) D contains a clique of size greater than $\Delta - \frac{5}{2}k$; and

(b) $|D| \le \Delta + \frac{3}{2}k + 2$.

Proof. By Lemma 2.5, D contains a $(\Delta - k + 1)$-critical dense subgraph H which is a clique Q joined to a graph H' where $\chi(H') \le \frac{3}{2}k$. Since $\chi(H) = \Delta - k + 1$, $|Q| = \chi(Q) > \Delta - \frac{5}{2}k$.

Every vertex in H has degree at least $\Delta - k$ in H since H is critical, and $|H| \le \Delta + 1$ since H has a dominating vertex. Thus $|E(H, D - H)| \le k(\Delta+1)$. Consider any $v \in D - H$. From the definition of a d-dense set, v has at least $\frac{3\Delta}{4}$ neighbors in D and $|D| \le \Delta + 4d + 1$. So, v is adjacent to at least $\frac{2\Delta}{3}$ vertices of H. Thus $|D - H| \le |E(H, D-H)| / \frac{2\Delta}{3} \le \frac{3}{2}k(\Delta+1)/\Delta < \frac{3}{2}k + 1$ (since $k \le \frac{2}{5}\Delta$). ■

Theorem 4.6 *For sufficiently large Δ and any $k \le \epsilon\sqrt{\Delta}$ (where ϵ is as in Theorem 1.3), the number of non-isomorphic $(\Delta - k + 1)$-critical graphs of maximum degree less than or equal to Δ is less than $2^{O(k^2)}$.*

Proof. By Theorem 1.3 and Lemma 2.5, if H is a $(\Delta - k + 1)$-critical graph of maximum degree less than or equal to Δ then H is a clique Q joined to a graph H', where $|H'| \le \frac{5}{2}k$. The number of choices for H', and hence for H, is $2^{O(k^2)}$. ■

Now we are ready to prove the main theorem of this section.

Proof of Theorem 4.1. We choose Δ_0 large enough so that Theorem 1.3 applies. Consider a d-dense decomposition of G, where $d = 1000\sqrt{\Delta}$. Reed[12] shows how to find such a decomposition in linear time. By Theorem 4.4, it is sufficient to test $(\Delta - k)$-colourability of each d-dense set D. We will show that we can test a dense set D in $O(\Delta^2)$ time. Since the number of edges in each dense set is of order $\Theta(\Delta^2)$, this will result in a total running time which is linear in $|E(G)|$.

Let D be a dense set. We repeatedly delete all the vertices in D with degree less than $\Delta - k$ to form D'. Fact 2.2 implies that $G(D')$ contains a $(\Delta - k + 1)$-critical graph iff $G(D)$ contains one.

Claim 4.7 *If $\chi(G(D)) > \Delta - k$ then a clique C of size at least $\Delta - \frac{5}{2}k - (10k^2 + 13k + 4)$ in D' can be found in $O(\Delta^2)$ time where C is dominating in D'.*

Proof. If $\chi(G(D)) > \Delta - k$ then D contains a $(\Delta - k + 1)$-critical graph and so does D'. Thus, D' contains a clique of size $\Delta - \frac{5}{2}k$. Let C be a such a clique. Every missing edge in D' must have an endpoint in $D' - C$. So the total number of missing edges in D' is at most $|D' - C| \times (|D'| - (\Delta - k)) \leq (\Delta + \frac{3}{2}k + 2 - (\Delta - \frac{5}{2}k)) \times (\Delta + \frac{3}{2}k + 2 - (\Delta - k)) = 10k^2 + 13k + 4$. Thus, if we let C be the set of vertices which are dominating in D', then C is a clique of size at least $\Delta - \frac{5}{2}k - (10k^2 + 13k + 4)$. ■

Since k is a constant, this finds a clique C of size $\Delta - O(1)$ in $O(\Delta^2)$ time.

Let $\mathcal{H}$ denote the set of all $(\Delta - k + 1)$-critical graphs and their supergraphs (on the same vertex set). Thus, G is not $(\Delta - k)$-colourable iff it contains one of the graphs in $\mathcal{H}$ as an induced subgraph. $|\mathcal{H}| = O(1)$ since, by Lemma 4.6, there are $O(1)$ non-isomorphic $(\Delta - k + 1)$-critical graphs, and each of them is missing $O(1)$ edges, and thus has $O(1)$ supergraphs.

Consider any $H \in \mathcal{H}$. We know that H is a clique C which dominates a graph Q, where Q has no dominating vertex. Thus, H is an induced subgraph of $D^{'}$ iff Q is an induced subgraph of $D' - C$. Since $D' - C$ has size $O(1)$, we can test whether it has Q as an induced subgraph in $O(1)$ time using exhaustive search.

Thus, after finding C, we can test whether D' contains any member of $\mathcal{H}$ in time $O(1)$. ∎

5 Graphs without large complete subgraphs

The relation between the maximum degree, chromatic number and clique number of a graph, has raised many interesting problems. Finding the best possible upper bound for the chromatic number of a graph G in the terms of the maximum degree $\Delta(G)$ when G does not contain a $(\chi(G) - k)$-clique as a subgraph, is one of them. The cases $k = 0$ and $k = 1$ are discussed individually in "Graph Colouring Problems" [6] as follows:

Problem 5.1 *Does there exist a Δ-chromatic graph without a Δ-clique as a subgraph and of maximum degree Δ for any value of $\Delta \geq 9$?* (This is problem 4.8 of [6]).

and

Problem 5.2 *Find the best possible upper bound in terms of $\Delta(G)$ for the chromatic number $\chi(G)$ of a graph G not containing a $(\chi - 1)$-clique.* (This is a slight rephrasing of problem 4.7 of [6]).

Reed (Theorem 1.2) proved that every graph with $\chi \geq \Delta \geq 10^{14}$ contains a Δ-clique, and thus $\chi \leq \Delta - 1$ is the best possible upper bound on χ for $k = 0$. Thus the answer to Problem 5.1 is 'No' for sufficiently large Δ.

For $k = 1$, Brooks' Theorem implies that $\Delta \geq \chi$, which is not the best possible lower bound. We show that Theorems 1.2, 3.1 and 3.2 yield that $\Delta \geq \chi + 3$ is the best possible lower bound for Δ in terms of χ when the graph does not contain a $(\chi - 1)$-clique. Thus the solution for the open problem 4.7 in [6] for $\Delta \geq \Delta_0 + 3$ is $\Delta \geq \chi + 3$.

Theorem 5.3 *Consider a graph G of maximum degree $\Delta \geq \Delta_0 + 3$. If G does not contain a $(\chi - 1)$-clique, then $\Delta \geq \chi + 3$ and this lower bound is best possible.*

Proof. We prove $\chi \leq \Delta - 3$. For $\chi \geq \Delta$, Theorem 1.2 implies that G contains a clique of size $\Delta = \chi$.

For $\chi \geq \Delta - 1$, note that a $(\Delta - 4)$-clique joined to a C_5 contains a $(\Delta - 2)$-clique. Thus, Theorem 3.1 implies that G contains a clique of size $\Delta - 2 = \chi - 1$.

Both graphs shown in figure 1 contain a triangle. So, if $\chi \geq \Delta - 2$, Theorem 3.2 implies that G contains a clique of size $\Delta - 3 = \chi - 1$.

To show that this is the best possible, let G be a $(\Delta - 9)$-clique joined to the join of two C_5's (figure 5). It can be seen that $\chi(G) = \Delta - 3$. Also, there is no 5-clique in the join of two C_5's, and so $\omega(G) = \Delta - 5 = \chi - 2$. ■

We consider the following more general problem:

Open Problem 5.4 *Consider a graph G of maximum degree Δ. Find the best possible upper bound in terms of Δ for the chromatic number $\chi(G)$ of a graph G not containing a $(\chi - k)$-clique.*

Reed [12] conjectures that for every graph G of maximum degree Δ, and with maximum clique-size ω, $\chi(G) \leq \lceil (\Delta + 1 + \omega)/2 \rceil$. If true, this

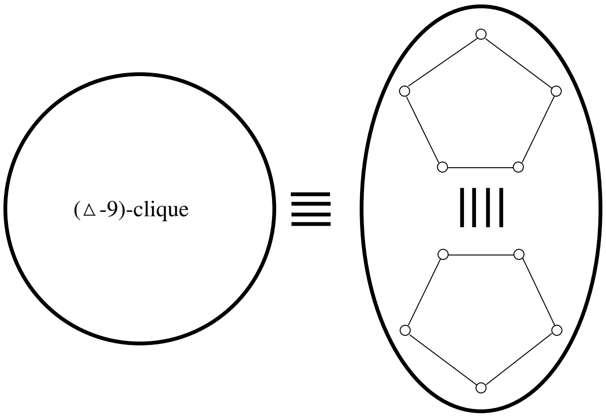

Figure 5:

would imply that every graph as in Problem 5.4 has chromatic number at most $\Delta - k + 1$. In the same paper, Reed provides an example of a graph with $\chi = \Delta + 1 - \ell$ and $\omega \leq \Delta - 2\ell + \ell^{4/5}$, for any sufficiently large ℓ. Setting $k = \ell - \ell^{4/5}$ this implies that for large k, there is no upper bound for Problem 5.4 that is smaller than $\Delta - k - o(k)$. Here, we see that the results of this paper are helpful for small k; in particular they answer the problem for $k = 2$ and Δ sufficiently large.

We first note that the structure of the graph in figure 5 can be generalized to show that the upper bound in Problem 5.4 must be at least $\Delta - (2k+1)$.

Theorem 5.5 *For all $k \geq 0$ and all $\Delta \geq 5(k+1)$, there is a graph G with $\chi = \Delta - (2k+1)$ where G does not contain a $(\chi - k)$-clique.*

Proof. Suppose that $C_{k+1,5}$ is the join of $k+1$ disjoint C_5's. Let G be a $(\Delta+1-5(k+1))$-clique joined to $C_{k+1,5}$. $\chi(G) = \Delta - 5(k+1) + 1 + 3(k+1) = \Delta - (2k+1)$ and $\omega(G) = \Delta - 5(k+1) + 1 + 2(k+1) \leq \Delta - 3k - 2 < \chi(G) - k$. ■

On the other hand, by examining the graphs listed in Theorems 3.1, 3.2, 3.3 and 3.4, we see that every graph G with $\chi(G) \geq \Delta(G) - 4$, and with $\Delta(G)$ sufficiently large, has a clique of size at least $\chi(G) - 2$. Therefore, for $k = 2$ and for Δ sufficiently large, the best possible upper bound sought for in Problem 5.4 is indeed $\chi \leq \Delta - 5$.

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