Chromatic Neighbourhood Sets

Michael Molloy
Department of Computer Science
University of Toronto
Toronto, Canada

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Abstract

For each vertex v in a graph G, we denote by χ_v the chromatic number of the subgraph induced by its neighbourhood, and we set $\chi_N(G) = \{\chi_v : v \in V(G)\}$. We characterize those sets X for which there exists some G of prescribed size with $X = \chi_N(G)$, and prove a related conjecture of Fajtlowicz. We also discuss those graphs which are extremal with respect to $\chi_N(G)$.

For each vertex v in a graph G, we denote by N_v the subgraph induced by the neighbourhood of v, where by *neighbourhood* we mean the set of vertices adjacent to v and we do not include v. We denote by χ_v the chromatic number of N_v. We define the *chromatic neighbourhood set* of G, $\chi_N(G)$ to be the set $\{\chi_v : v \in V(G)\}$.

The set $\chi_N(G)$ was introduced in [3] in which there are a number of GRAFFITI conjectures concerning it. In this paper, we characterize all sets of integers, X, which are the chromatic neighbourhood sets of some graph G on n vertices. As a corollary, we prove the GRAFFITI conjecture [3] that for all G, $|\chi_N(G)| \leq \lceil \frac{|G|}{2} \rceil$. We then examine the graphs G for which $\chi_N(G)$ is extremal with respect to our characterization.

The author's hope is that these preliminary results will promote further study of chromatic neighbourhood sets and related graph parameters such as the *chromatic neighbourhood sequence* $(\chi_v)_{v \in G}$.

As usual, we use $\overline{G}$ to denote the complement of a graph G. A $\overline{G}$-*neighbour* of a vertex v is a neighbour in $\overline{G}$, i.e. a non-neighbour in G. Similarly, for a set of vertices H we sometimes use $\chi_G(H)$ or the *G-chromatic number* of H to refer to the chromatic number of the subgraph of G induced by H, thus distinguishing between $\chi_G(H)$ and $\chi_{\overline{G}}(H)$. When there is no need to specify G, we simply say $\chi(H)$, a slight abuse of notation. A graph G is said to be *k-critical* if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph $H \subset G$. It is well-known (and trivial to show) that if $\chi(G) \geq k$ then G must have a k-critical (not neccessarily proper) subgraph. For any set X and element $x \in X$, we use $X \backslash x$ to denote the set obtained by removing x from X. If $x \notin X$ then $X \backslash x = X$. The *size* of a graph is the number of vertices. The *size* of a matching is the number of edges, i.e. twice the number of vertices. A vertex is *saturated* by a matching if it is the endpoint of one of the edges of the matching. A set of vertices is *saturated* if every member of the set is saturated. For any other graphical definitions, we refer the reader to [6].

1 A Simple Characterization

Our first result is the following simple neccessary and sufficient condition for a sequence X to be the chromatic neighbourhood sequence of a graph G of size n:

Theorem 1 *Suppose $X = \{d_1, ..., d_t\}$ where $0 \leq d_1 < ... < d_t$. Then X is the chromatic neighbourhood set of some graph G of size n if and only if $t \leq n - d_t$.*

That this condition is neccessary follows from the following lemma:

Lemma 1 *Suppose $\chi_N(G) = \{d_1, ..., d_t\}$ where $0 \leq d_1 < ... < d_t$. Then $t \leq |G| + d_t - 2\chi(G) + 2$.*

Proof Set $n = |G|$. First note that $\overline{G}$ has no matching of size $n - \chi(G) + 1$, since if it did we could obtain a $\chi(G) - 1$ colouring of G by considering the vertices of each edge from the matching to be a colour class of size 2, and each vertex not saturated by the matching to be a colour class of size 1. Therefore, by Berge's variation on Tutte's Theorem (see eg. [2]) there is some $X \subseteq V(G)$ such that $\overline{G} - X$ has at least $|X| + 2\chi(G) - n$ odd components.

By taking a vertex from each component of $\overline{G} - X$, we form an independent set in $\overline{G}$ and hence a clique in G. Therefore, each vertex in one of these components lies in a $(|X| + 2\chi(G) - n)-$clique of G, and so for each $v \notin X$, $\chi_v \geq |X| + 2\chi(G) - n - 1$. Therefore at most $|X|$ elements of $\chi_N(G)$ are less than $|X| + 2\chi(G) - n - 1$ and so

$$\begin{aligned} t &\leq |X| + |\chi_N \cap \{i | i \geq |X| + 2\chi(G) - n - 1\}| \\ &\leq |X| + d_t - (|X| + 2\chi(G) - n - 1) + 1 \\ &\leq d_t - 2\chi(G) + n + 2 \end{aligned}$$

□

Remark: The idea of using a matching in $\overline{G}$ to produce a vertex colouring of G is not new. For example, it appears in [3] and [4].

The proof of Theorem 1 now follows:

Proof of Theorem 1:

That this condition is neccessary follows from Lemma 1 and the fact that $\chi(G) \geq d_t + 1$. That it is sufficient follows from the following construction due to Erdős and Fajtlowicz[3].

Take a d_t-clique, and add t vertices $v_1, ..., v_t$ where v_i is adjacent only to d_i vertices of the clique. (In the case that $n > t + d_t$, simply add some twins of v_1, i.e. vertices with the same neighbourhood as v_1.) □

As a corollary, we obtain a proof of the aforementioned GRAFFITI conjecture:

Corollary 1 *For all G, $|\chi_N(G)| \leq \lceil \frac{|G|}{2} \rceil$.*

Proof As usual, we let $n = |G|$, and let $\chi_N(G) = \{d_1, ..., d_t\}$ where $d_1 < ... < d_t$.

If $d_1 \geq 1$ then $d_t \geq t$, and so by Theorem 1, $t \leq \frac{n}{2}$.

If $d_1 = 0$ and n is odd, then $d_t \geq t-1$ and so by Theorem 1, $t \leq \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.

The only remaining case is $d_1 = 0$ and n is even. In this case, we note that since $d_1 = 0$, G has some isolated vertices, and we let G' be the graph formed by deleting all isolated vertices from G. Of course, $0 \notin \chi_N(G')$, and so G' falls under the first case we examined which implies that $|\chi_N(G')| \leq \frac{|G'|}{2}$. In fact, $\chi_N(G') = \chi_N(G) \backslash 0$ and so we have $t - 1 = |\chi_N(G')| \leq \frac{|G'|}{2} \leq \frac{n-1}{2}$. Since $t - 1$ is an integer and $\frac{n-1}{2}$ is not, we have $t - 1 \leq \frac{n}{2} - 1$, i.e. $t \leq \frac{n}{2}$. $\Box$

2 The Extremal Graphs

Next, we describe those graphs G, which yield one of the extremal values of $\chi_N(G)$, i.e. one with $|G| = t + d_t$. In paticular, we prove the following:

Theorem 2 *If d_t is the largest member of $\chi_N(G)$, and $|G| = |\chi_N(G)| + d_t$, then G contains a $(d_t + 1)$-clique.*

Thus, all such graphs are simple variations of the construction of Erdős and Fajtlowicz described in the proof of Theorem 1. I.e., they can be formed by taking a d_t-clique, K, and adding vertices $v_1, ..., v_t$ such that v_t is adjacent to all of K, each v_i is adjacent to at most d_i vertices of K, and for each i, $\chi_{v_i} = d_i$. Unfortunately, we can not hope for a much cleaner characterization of these graphs since as we will show in Section 4, it is NP-complete to determine whether a graph G satisfies $|G| = |\chi_N(G)| + d_t$.

We will need a few lemmas, which we list first, delaying most of their proofs until the next section. The first one, a chromatic variant of Tutte's Theorem, and its corollary may be of independent interest.

Lemma 2 *For any graph G with $\chi(G) > \frac{|G|}{2}$, there is some $X \subseteq V(G)$ such that $\overline{G} - X$ has components $C_1, C_2, ..., C_t$, $t > |X|$, and where*

(a) for each $1 \leq i \leq t$, $\chi_G(C_i) \leq \lceil \frac{|C_i|}{2} \rceil$,

(b) for each $1 \leq i \leq |X| + 1$, $|C_i|$ is odd and $\chi_G(C_i) = \lceil \frac{|C_i|}{2} \rceil$,

(c) $\sum_{i=1}^{t} \chi_G(C_i) = \chi(G)$.

In what follows, we only use the following corollary of Lemma 2.

Corollary 2 *If H is a k-critical graph, and $|H| \leq 2k - 2$, then $\overline{H}$ is disconnected.*

Proof Taking $X \subset V(H)$ as guaranteed by Lemma 2, $\chi(H - X) = \chi(H)$, and so by the criticality of H, $X = \emptyset$. Since $\chi(H) > \lceil \frac{|H|}{2} \rceil$, $\overline{H}$ has at least 2 components, by Lemma 2(a). $\square$

Before proceeding with the next lemmas, we introduce some more notation. For each subgraph $H \subseteq G$, we set $\chi_N(H) = \{\chi_v | v \in H\}$. For each $v \in H$, we denote by χ_v^H the chromatic number of the subgraph induced by $N(v) \cap H$, and we set $\chi_N^H(H) = \{\chi_v^H | v \in H\}$. Note that if H^* is a graph isomorphic to H, then $\chi_N(H^*) = \chi_N^H(H)$.

As usual, the *join* of H_1 and H_2 is the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2) \cup (V(H_1) \times V(H_2))$. We say that subgraphs $H_1, H_2 \subseteq G$ are *joined* if $V(H_1) \times V(H_2) \subseteq E(G)$.

Lemma 3 *If $H \subseteq G$ is joined to a subgraph of chromatic number $r+1$, and $\chi(H) = d_t - r$ then $|H| - |\chi_N(H) \backslash d_t| \geq d_t - r$.*

Lemma 4 *If a $(d_t - r)$-critical subgraph $H \subseteq G$ satisfies*

(a) H is joined to a subgraph Y of chromatic number $(r+1)$,

(b) $|H| \leq 2(d_t - r) - 1$, and

(c) $|H| - |\chi_N(H) \backslash d_t| = d_t - r$,

then H is a $(d_t - r)$-clique.

And now, finally, with these lemmas in hand we can prove Theorem 2:

Proof of Theorem 2: Clearly, we can restrict our attention to the case $d_t > 0$.

Choose v such that $\chi_v = d_t$ and choose H to be any d_t-critical subgraph of N_v. We will apply Lemma 4 with $Y = v$ and $r = 0$ to show that H is a d_t-clique and so $v \cup H$ is a $(d_t + 1)$-clique, thus proving the theorem. Clearly H meets condition (a) of Lemma 4, and so we only need to verify conditions (b) and (c).

Condition (b): Obviously $|\chi_N(G)| \leq d_t + 1$, and if equality holds then $0 \in \chi_N(G)$. Furthermore, since $v \notin H$, $|H| \leq |G| - 1$, and if $0 \in \chi_N(G)$, then G has an isolated vertex which is also not in H and so $|H| \leq |G| - 2$. Therefore $|H| \leq 2d_t - 1$.

Condition (c): Since $v \notin G-H$ and $\chi_v = d_t$ we have $|G|-|H| \geq |\chi_N(G)| - |\chi_N(H)\backslash d_t|$. Therefore since $|G| = |\chi_N(G)| + d_t$ we have $|H| - |\chi_N(H)\backslash d_t| \leq d_t$. On the other hand, H is joined to v, and so by applying Lemma 2 with $r = 0$ we have $|H| - |\chi_N(H)\backslash d_t| \geq d_t$. Therefore, $|H| - |\chi_N(H)\backslash d_t| = d_t$. $\square$

It only remains to prove Lemmas 2, 3 and 4 which we will do in the next section.

3 The Rest of the Proofs

Proof of Lemma 2: The proof is by induction on $|G|$. The case $|G| = 1$ is trivially satisfied by taking $X = \emptyset$.

If $\chi(G) = \lceil \frac{|G|}{2} \rceil$, then $|G|$ must be odd and so we can take $X = \emptyset$. Thus we can assume $\chi(G) > \lceil \frac{|G|}{2} \rceil$. Also, if $\overline{G}$ is disconnected then it is straightforward to apply the inductive hypothesis to the components of $\overline{G}$, and so we can assume $\overline{G}$ is connected.

With these assumptions, $\overline{G}$ has no matching of size at least $\lfloor \frac{|G|}{2} \rfloor$, as otherwise we can use the edges of the matching as colour classes to obtain a $\lceil \frac{|G|}{2} \rceil$ colouring. Therefore, by Tutte's Theorem, there exists a non-empty set X such that the number of odd components in $\overline{G} - X$ is greater than $|X|$. Given such an X, we denote the components of $\overline{G} - X$ by $C_1, ..., C_t$, where the s odd components with $\chi_G(C_i) \geq \lceil \frac{|C_i|}{2} \rceil$ are listed first, i.e. as $C_1, ..., C_s$. Choose such an X to first maximize the sum of the G-chromatic numbers of the components of $\overline{G} - X$, then to maximize $s - |X|$, and lastly to maximize $|X|$. We will prove that X satisfies the assertions of our lemma.

First, we verify assertion (a) of the lemma:

Claim 1: For $1 \leq i \leq t$, $\chi_G(C_i) \leq \lceil \frac{|C_i|}{2} \rceil$.

Proof: Suppose that there is some C_i with $\chi_G(C_i) > \lceil \frac{|C_i|}{2} \rceil$. Since $X \neq \emptyset$, $|C_i| < |G|$, and so by the induction hypothesis, there is some $X^{'} \subset C_i$, where the components of $C_i - X^{'}$ are $D_1, ..., D_r$, $r > |X^{'}|$ and such that for each $1 \leq j \leq |X^{'}| + 1$, $|D_j|$ is odd, and $\sum_{j=1}^{r} \chi_G(D_j) = \chi_G(C_i)$. Thus, by setting $X_1 = X \cup X^{'}$, we violate the maximality of $|X|$.

Our next claim will aid us in our proof of assertions (b) and (c).

Claim 2: For each $x \in X$, and each $1 \leq i \leq s$, if x has a $\overline{G}$-neighbour in

C_i, then $\chi_G(C_i \cup \{x\}) = \chi_G(C_i)$.

Proof: Suppose otherwise. Since $t > |X|$ we have $|C_i \cup \{x\}| < |G|$, and furthermore, since $i \leq s$, we have $\chi_G(C_i \cup \{x\}) > \lceil \frac{|C_i|}{2} \rceil = \frac{|C_i \cup \{x\}|}{2}$. Therefore, by the induction hypothesis, there is some $X' \subset C_i \cup \{x\}$, where the components of $(C_i \cup \{x\}) - X'$ are $D_1, ..., D_r$, $r > |X'|$, and such that for each $1 \leq j \leq |X'|$, $|D_j|$ is odd, and $\sum_{j=1}^{r} \chi_G(D_j) = \chi_G(C_i \cup \{x\})$. Furthermore, since C_i is connected, $X' \backslash x \neq \emptyset$. Therefore, by setting $X_1 = X \cup X'$, we violate the maximality of $|X|$.

Next we prove assertion (b).

Claim 3: $s \geq |X| + 1$.

Proof: Suppose $s \leq |X|$. For every odd component $C_i, i > s$, we have $\chi_G(C_i) < \lceil \frac{|C_i|}{2} \rceil$.

Set $\mathcal{C} = \{C_1, ..., C_s\}$ and define B to be the bipartite graph with bipartition $(X, \mathcal{C})$, where $(x, C_i) \in E(B)$ iff x has a $\overline{G}$-neighbour in C_i. If B has a matching which saturates every $C_i \in \mathcal{C}$, then letting X_B be the s vertices of X saturated by the matching and applying Claim 2, we can colour all vertices in $C_1 \cup ... \cup C_s \cup X_B$ with $\sum_{i=1}^{s} \chi_G(C_i) = \frac{1}{2}|C_1 \cup ... \cup C_s \cup X_B|$ colours. By letting each vertex in $X - X_B$ be a colour class of size 1, we can colour all vertices in $C_{s+1} \cup ... \cup C_{|X|+1} \cup (X - X_B)$ using $|X - X_B| + \sum_{i=s+1}^{|X|+1} \chi_G(C_i) < \frac{1}{2}|C_{s+1} \cup ... \cup C_{|X|+1} \cup (X - X_B)|$. Finally, for each $i > |X| + 1$ we can colour the vertices of C_i using at most $\frac{1}{2}|C_i|$ colours. Therefore $\chi(G) < \frac{1}{2}|G|$ which contradicts the prerequisites of our lemma.

Therefore B does not have a matching which saturates $\mathcal{C}$. Thus, by Hall's Theorem, there is some collection of components in $\mathcal{C}$, say $C_1, ..., C_r$ such that the union of their $\overline{G}$-neighbourhoods in X is less than r. Setting X_1 to be this subset of X, clearly the sum of the G-chromatic numbers of the components of $\overline{G} - X_1$ is at least the corresponding sum for $\overline{G} - X$ and $C_1, ..., C_r$ are all components of $\overline{G} - X_1$. Therefore this violates the maximality of $s - |X|$.

And now finally, we can prove assertion (c).

Claim 4: $\sum_{i=1}^{t} \chi_G(C_i) = \chi(G)$.

Proof: Clearly, $\sum_{i=1}^{t} \chi_G(C_i) \geq \chi(G)$. Our task is to use Claim 2 to show that we can colour each $\overline{C_i}$ such that we can then colour the vertices in X without using any new colours. Thus, if we define $\mathcal{C}$ and B as in the proof of

Claim 3, it would be enough if B had a matching which saturates all of X.

If B does not have such a matching, then by Hall's Theorem there is some $Y \subseteq X$ such that $|N_B(Y)| < |Y|$. Since $|\mathcal{C}| > |X|$ and $\overline{G}$ is connected, $Y \neq X$. Setting D to be the union of Y with all components of $\overline{G} - X$ to which Y is adjacent, if $\chi_G(D) > \chi_G(D - Y)$, then by setting $X_1 = X - Y$, we violate the maximality of $\sum \chi_G(C_i)$. Therefore, $\chi_G(D) = \chi_G(D - Y)$ and so removing Y and the components in D from B, to yield $B^{'}, X^{'}$ and $\mathcal{C}^{'}$, it is enough to find a matching in $B^{'}$ which saturates $X^{'}$. Repeating this argument until either we find such a matching or we reduce $X^{'}$ to the empty set proves the claim. □

Proof of Lemma 3: For each $v \in H$, set $x_v = \chi_v - (r+1)$. Clearly, $\chi_v^H \leq x_v \leq \chi(H) - 1$. It is now straightforward to apply the argument used to prove Lemma 1 to show that $|\{x_v : v \in H\}| \leq |H| + (\chi(H) - 1) - 2\chi(H) + 2 = |H| - d_t + r + 1$, and that furthermore, equality holding requires that $x_v = \chi(H) - 1$ for some $v \in H$, i.e. that $d_t \in \chi_N(H)$. Since $|\chi_N(H)| = |\{x_v : v \in H\}|$, we have $|H| - |\chi_N(H) \backslash d_t| \geq d_t - r$. □

Proof of Lemma 4: The proof is by induction on $|H|$. The case $|H| = 1$ is trivial. Thus we can assume $|H| > 1$ which implies that H is not 1-critical and so $d_t - r > 1$.

Claim: $|H| \leq 2(d_t - r) - 2$

Proof: Since H is critical, for each $v \in H$, $\deg_H(v) \geq 1$, and so, since H is joined to Y, $\chi_v \geq r + 2$. Therefore, every member of $|\chi_N(H) \backslash d_t$ lies in the range $[r+2, d_t - 1]$ and so $|\chi_N(H) \backslash d_t| \leq (d_t - 1) - (r+2) + 1 = d_t - r - 2$. The claim now follows from the prerequisite $|H| - |\chi_N(H) \backslash d_t| = d_t - r$.

By our Claim and Corollary 2, $\overline{H}$ is not connected, and so H is the join of H_1 and H_2 where each H_i is k_i-critical for some $k_1 + k_2 = d_t - r$. At least one H_i has size at most $2k_i - 1$; suppose it is H_1. We will now show that $|H_2| \leq 2k_2 - 1$ as well.

H_1 is joined to the $(k_2 + r + 1)$-chromatic subgraph $Y \cup H_2$, and so by Lemma 3,

$$|H_1| - |\chi_N(H_1) \backslash d_t| \geq k_1 \tag{1}$$

Also, for each $v \in H_2$, $\chi_v \geq k_1 + r + 1$. Therefore, $|\chi_N(H_2) \backslash d_t| \leq$

$(d_t - 1) - (k_1 + r + 1) + 1 = k_2 - 1$, and so

$$\begin{aligned}
|H_2| &= |H| - |H_1| \\
&= |\chi_N(H)\backslash d_t| + d_t - r - |H_1| \\
&\leq |\chi_N(H_2)\backslash d_t| + d_t - r + |\chi_N(H_1)\backslash d_t| - |H_1| \\
&\leq (k_2 - 1) + d_t - r + (-k_1) \\
&= 2k_2 - 1
\end{aligned}$$

Thus, by symmetry (1) holds for H_2 as well, i.e. $|H_2| - |\chi_N(H_2)\backslash d_t| \geq k_2$. But, $k_1 + k_2 = |H| - |\chi_N(H)\backslash d_t| \geq |H_1| - |\chi_N(H_1)\backslash d_t| + |H_2| - |\chi_N(H_2)\backslash d_t|$, and so $|H_i| - |\chi_N(H_i)\backslash d_t| = k_i$ for $i = 1, 2$. It follows by the inductive hypothesis that each H_i is a k_i-clique, and so H is a $(d_t - r)$-clique. □

4 The Complexity of Recognizing Extremal Graphs

Here we note the following:

Theorem 3 *It is NP-complete to decide whether a graph G satisfies $|G| = |\chi_N(G)| + d_t$ where d_t is the largest member of $\chi_N(G)$.*

Proof We will reduce from 3-COLOURABILITY-OF-PLANAR-GRAPHS, which is well-known to be NP-complete as shown by Stockmeyer[5].

Suppose that we wish to determine whether a particular planar graph G of size $n \geq 7$, is 3-colourable. We will construct a graph H of size $\mathrm{O}(n^2)$ such that G is 3-colourable iff H satisfies $|H| = \chi_N(H) + d_t$, thus proving our theorem.

To construct H, we begin with a clique K of size n^2. We label the vertices of G as $v_4, ..., v_{n+3}$ and we join each v_i to i vertices in K such that no two vertices in G have a common neighbour in K (this is possible for $n \geq 7$). Add a new vertex, v_3 where $N(v_3) = G$. Finally, add vertices $v_1, v_2, v_{n+4}, v_{n+5}, ..., v_{n^2}$ where each new v_i is adjacent to precisely i vertices in K, and to no vertices outside of K. Since $\chi(G) \leq 4$, for each $4 \leq i \leq n+3$, $\chi_{v_i} = i$, and clearly for $i = 1, 2, n+4, n+5, ..., n^2$ we also have $\chi_{v_i} = i$. Thus $\chi_N(H) = \{1, ..., n^2\}$ iff $\chi_{v_3} = 3$, i.e. iff $\chi(G) = 3$, as required. □

5 One Last Application

GRAFFITI [3] also conjectures something stronger than Corollary 1. Given a graph G, define $R \subset \overline{G}$ to be the graph defined by $V(R) = V(G)$ and $e \in E(R)$ iff $\chi(G+e) > \chi(G)$.

Conjecture: *If G is connected then $2|\chi_N(G)|$ is no greater than the number of components of R.*

This conjecture was recently disproven by Bollobás and Riordan [4]. However, we note that Theorem 2 yields the following special case:

Corollary 3 *If G has no isolated vertices and $|\chi_N(G)| \geq \frac{|G|}{2}$ then R has no edges.*

Proof As usual, set $\chi_N(G) = \{d_1, ..., d_t\}$, where $d_1 < ... < d_t$. Since G has no isolated vertices, $d_1 \geq 1$, and so as in the proof of Corollary 1, it is straightforward to show from Theorem 1 that $t \leq \lfloor\frac{|G|}{2}\rfloor$. Therefore, if $t \geq \frac{|G|}{2}$ then $|G|$ must be even, say $2k$, and it is straightforward to modify the proof of Lemma 1 to show that $t \leq |G| + d_t - 2\chi(G) + 1$. Applying $\chi(G) \geq d_t + 1$ and $d_t \geq t$ (since $d_1 \geq 1$) we obtain $k = t = d_t = \chi(G) - 1$. Thus $\chi_N(G) = \{1, ..., k\}$.

Therefore, Theorem 2 implies that G has a $(k+1)$-clique K and so we can label $L = G - K$ as $v_1, ..., v_{k-1}$ where each v_i has at most i neighbours in K.

Consider any edge $e \notin E(G)$. We will show that $\chi(G+e) = k+1$, thus establishing that $e \notin R$, i.e. that R has no edges. Define B to be the bipartite graph with bipartition (K, L) where an edge between K and L is in B iff it is not in $G+e$. It is a straightforward application of Hall's Theorem (see [6]) to show that B has a matching which saturates L. Indeed, for any $X \subseteq L$, at least one vertex in X has degree at least $|X|$ in B, and so $|N_B(X)| \geq |X|$. Therefore, after assigning colour i to v_i for each $1 \leq i \leq k-1$, we can extend the colouring to a $(k+1)$-colouring of $G+e$. $\square$

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