Colouring a Graph Frugally*

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Abstract

We prove that any graph with maximum degree Δ sufficiently large, has a proper vertex colouring using $\Delta+1$ colours such that each colour class appears at most $\log^5 \Delta$ times in the neighbourhood of any vertex. We also show that for $\beta \geq 1$, the minimum number of colours required to colour any such graph so that each vertex appears at most β times in the neighbourhood of any vertex is $\theta(\Delta+\Delta^{1+1/\beta}/\beta)$, showing in particular that when $\beta = \log \Delta/\log\log \Delta$, such a colouring cannot always be achieved with $O(\Delta)$ colours. We also provide a polynomial time algorithm to find such a colouring. This has applications to the total chromatic number of a graph.

1 Introduction

It is well known that any graph with maximum degree Δ can be properly $(\Delta+1)$-coloured using a simple greedy algorithm. In this paper we show

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that in fact such a graph can be properly $(\Delta+1)$-coloured so that no colour appears more than $\mathrm{poly}(\log\Delta)$ times in the neighbourhood of each vertex.

We say that a vertex colouring of a graph is *β-frugal* if no vertex has more than β members of any colour class in his neighbourhood. Our main result is the following:

Theorem 1 *Every graph G with maximum degree $\Delta \geq \Delta_0 = \mathrm{e}^{10^7}$ has a $\lceil \log^5 \Delta \rceil$-frugal $(\Delta+1)$-colouring.*

Our proof is probabilistic, and makes use of the Lovasz Local Lemma. The proof can be made constructive, providing an $\mathrm{O}(n^3 \log^{\mathrm{O}(1)} n)$ randomized algorithm, and a polytime deterministic algorithm to find such a colouring.

In [5] we make use of these results to find a total colouring of any graph with maximum degree Δ sufficiently large using at most $\Delta + \mathrm{poly}(\log\Delta)$ colours.

The motivation behind the approach we take is simple. First, we note that it is enough to prove the case where G is Δ-regular. We then set $k = k_\Delta = \lceil \log^2 \Delta \rceil$, we set $\ell = \ell_\Delta = \lceil \frac{\Delta}{k} \rceil$, and we attempt to partition $V = V(G)$ into sets $S_1, \ldots, S_\ell$ such that

(a) for each vertex v, and each i in $\{1, \ldots, \ell\}$, $|N(v) \cap S_i| \leq k$, and

(b) each set S_i has a k-colouring.

(Note that by Brooks' Theorem, if (a) holds then so does (b) as long as $k > 2$ and no S_i induces a clique.)

If we could find the desired partition, then we could combine the colourings of (b) to obtain a $k\ell \leq \Delta + k$ colouring of G in which each colour appears at most k times in the neighbourhood of any vertex (by (a)). With a little more care, we could actually create a $\Delta+1$ colouring which meets the stipulations of Theorem 1.

In proving the theorem, we attempt to implement this straightforward idea. Unfortunately, a number of complications arise. We present the details in Section 2.

In Section 3, we show that our result is nearly best possible, by showing that for any $t \geq 1$, and arbitrarily large Δ there exists a Δ-regular graph G such that for any $t\Delta$ colouring of G there is a vertex v with at least $\log\Delta/\log\log\Delta$ members of one colour class in his neighbourhood. We do this by showing that for $\beta \geq 1$, the number of colours required to assign a β-frugal colouring to any graph with maximum degree Δ is at least $\Delta^{1+1/\beta}/2\beta$. On the other hand, we show that we can find such a colouring with $\max(\beta\Delta, \mathrm{e}^3\Delta^{1+1/\beta}/\beta)$ colours.

In Section 4, we provide efficient algorithms to find the colourings guaranteed by these results.

Several times in our work, we make use of the Lovász Local Lemma [3], which we present now. Suppose that $\mathcal{A} = A_1, A_2, \ldots, A_n$ is a list of random events. We say that a directed graph G is a *dependency graph* of $\mathcal{A}$ if for each i, A_i is mutually independent of $\{A_j | (i,j) \notin E(G), j \neq i\}$.

The Local Lemma (General Form) *Suppose $\mathcal{A} = A_1, A_2, \ldots, A_n$ is a list of random events and G is a dependency graph of $\mathcal{A}$. Suppose further that there exist $x_1, \ldots, x_n > 0$ such that for $1 \leq i \leq n$,*

$$\mathbf{Pr}(A_i) \leq x_i \prod_{(i,j)\in E(G)} (1-x_j).$$

Then $\mathbf{Pr}(\cup_{i=1}^{n}\bar{A}_i) > 0$.

We will usually use the following special case:

The Local Lemma (Symmetric Form) *Suppose* $\mathcal{A} = A_1, A_2, \ldots, A_n$ *is a list of random events and* G *is a dependency graph of* $\mathcal{A}$*. Suppose further that there exist* $p, d > 0$ *such that for* $1 \leq i \leq n$, $\mathbf{Pr}(A_i) \leq p$, $deg_G(i) \leq d$ *and* $ep(d+1) < 1$*. Then* $\mathbf{Pr}(\cup_{i=1}^{n}\bar{A}_i) > 0$.

Note that the symmetric form follows from the general form by setting $x_i = 1/(d+1)$ for each i.

We will also make use of the following bounds on the tails of the binomial distribution (see for example [4]):

The Chernoff Bounds *Let* $B(n,p)$ *be the sum of* n *independent Bernoulli variables, each occuring with probability* p*. Then for any* $a > 0$ *we have the following:*

$$\mathbf{Pr}(B(n,p) - np > a) < \mathrm{e}^{-a^2/2np},$$

and

$$\mathbf{Pr}(B(n,p) - np < -a) < \mathrm{e}^{-a^2/3np}.$$

Throughout this paper, we use $A + B$ to denote the subgraph induced by the vertex set $A \cup B$.

2 Proceeding Carefully

Now we proceed with our proof of Theorem 1. To begin, we examine the properties of a random partition of $V(G)$ into ℓ sets. If each vertex is placed in each S_i with probability ℓ^{-1} and these choices are independent, then for a fixed vertex v we expect that the sizes of the sets $S_1 \cap N(v), \ldots, S_\ell \cap N(v)$ will remain between $k - \sqrt{\log \Delta}\sqrt{k}$ and $k + \sqrt{\log \Delta}\sqrt{k}$.

In fact, we can apply the Local Lemma to ensure that there is a vertex partitioning $S_1, \ldots, S_\ell$ such that for each $v \in V$ and $i \in \{1, \ldots, \ell\}$ we have $k - 3\sqrt{\log \Delta}\sqrt{k} \leq |S_i \cap N(v)| \leq k + 3\sqrt{\log \Delta}\sqrt{k}$. This is not quite what we want - our degrees are out by a fudge factor of $f = f_\Delta = 3\sqrt{\log \Delta}\sqrt{k}$ - so we cannot apply Brooks' Theorem.

So, we will actually construct a set U of vertices which contains about $\frac{3}{2}f\ell$ of the neighbours of each vertex in G, and then apply our original technique to obtain an ℓ partitioning of $G - U$ such that for each $v \in V$ and $i \in \{1, \ldots, \ell\}$ we have $k - 3f \leq |S_i \cap N(v)| \leq k - 2$.

To be precise, our first result is:

Lemma 1 *Let G be a Δ-regular graph where $\Delta \geq \Delta_0 = \mathrm{e}^{10^7}$, and set $k = \lceil \log^2 \Delta \rceil$, $\ell = \lceil \frac{\Delta}{k} \rceil$ and $f = \lceil 3\sqrt{\log \Delta}\sqrt{k} \rceil$. There exists a partition of $V(G)$ into sets $S_1, \ldots, S_\ell$ and U such that:*

1. *for each $v \in V(G)$, $f\ell \leq |N(v) \cap U| \leq 2f\ell$,*
2. *for each $v \in V(G)$, $i \in \{1, \ldots, \ell\}$, $k - 3f \leq |N(v) \cap S_i| \leq k - 2$.*

Proof We use the symmetric form of the Local Lemma. We partition the vertex set into $S_1, S_2, \ldots, S_l, U$ randomly, placing each vertex in U with probability $\frac{3}{2}f\ell/\Delta$, and in S_i with probability $(1 - \frac{3}{2}f\ell/\Delta)/\ell$, where the choice is made independently of the choices for the other vertices. Note that for each $v \in V(G), 1 \leq i \leq \ell$, $\mathbf{Exp}(|N(v) \cap U|) = \frac{3}{2} f \ell$ and $\mathbf{Exp}(|N(v) \cap S_i|) = k - \frac{3}{2} f + \rho$, for some $0 \leq \rho < 1$.

For each $v \in V(G), 1 \leq i \leq \ell$ let A_v be the event that v violates condition 1 and let $B_{v,i}$ be the event that v, i violates condition 2 of Lemma 1. By the Chernoff bounds,

$$\mathbf{Pr}(A_v) \leq 2\mathrm{e}^{-(\frac{1}{2} f \ell)^2 / 3(\frac{3}{2} f \ell)} \leq 2\mathrm{e}^{-\frac{\Delta}{\log \Delta}},$$

and

$$\mathbf{Pr}(B_{v,i}) \leq 2\mathrm{e}^{-f^2/3k} \leq \frac{2}{\Delta^3}.$$

Also, each event is independent of all but at most $2\Delta^2 \ell$ other events, and so as $\mathrm{e} \times (2/\Delta^3) \times (2\Delta^2 \ell + 1) < 1$, the probability that conditions 1 and 2 hold is positive. □

Suppose $\Delta = k\ell - r$, where $0 \leq r < k$. Note that each S_i clearly has a $k-1$ vertex colouring - we simply colour greedily.

It remains only to partition U into sets $U_1, \ldots, U_\ell$ such that

(a) for $1 \leq i \leq \ell - r$, $U_i + S_i$ has a k vertex colouring,
and for $\ell - r < i \leq \ell$, $U_i + S_i$ has a $k-1$ vertex colouring;

(b) no vertex sees more than $\log^5 \Delta - k$ vertices of any U_i.

The desired result then follows by combining the colourings described in (a).

We will construct the sets $U_1, \ldots, U_\ell$ recursively. Initially, $U_1 = \ldots = U_\ell = \emptyset$. In each iteration, we will add a stable set U_i' of $U - \cup_{j=1}^{\ell} U_j$ to U_i, each vertex of which sees at most $k-1$ vertices of $U_i + S_i$ for $i \leq \ell - r$, and at most $k-2$ vertices if $i > \ell - r$. This implies that we can greedily extend our colouring of each $U_i + S_i$ to $U_i^{'}$ so that after adding $U_i^{'}$ to U_i, (a) remains true. The crux of the proof is choosing these sets so that (b) holds when we are done.

To capture the key ideas, we informally describe the first iteration. To do so, we will examine a random procedure for choosing sets $U_1^{'}, \ldots, U_\ell^{'}$. For each vertex v, we let $I_v = \{i : |N(v) \cap S_i| \leq k - 1 \text{ and } |N(v) \cap S_i| \leq k - 2 \text{ if } i > \ell - r\}$. If we put v into $U_i^{'}$ then $i \in I_v$. We will put v into $\cup_{i \in I_v} U_i^{'}$ with probability f^{-2} and given that it goes into some $U_i^{'}$, each $U_i^{'}$ for $i \in I_v$ is equally likely. Thus for each $v \in V(G)$, $i \in \{1 \ldots, \ell\}$,

$$\mathbf{Pr}(v \in U_i^{'}) = \begin{cases} 0, & i \notin I_v \\ \frac{1}{f^2 |I_v|}, & i \in I_v \end{cases}$$

and this choice is made independently of the choice for any other vertex.

Let us investigate what the random sets $U_1^{'}, \ldots, U_\ell^{'}$ look like. First note that for any vertex $u \in U$ we have $f\ell \leq |N(u) \cap U| \leq 2f\ell$, and so

$$\Delta - 2f\ell \leq \sum_{i=1}^{\ell} |N(u) \cap S_i| \leq \Delta - f\ell$$

but

$$\sum_{1 \leq i \leq \ell - r} k + \sum_{\ell - r < i \leq \ell} k - 1 = \Delta$$

and so

$$\sum_{1 \leq i \leq \ell - r, i \in I_u} k - |N(u) \cap S_i| + \sum_{\ell - r < i \leq \ell, i \in I_u} k - 1 - |N(u) \cap S_i| \geq f\ell.$$

Furthermore, for each i, $|N(u) \cap S_i| \geq k - 3f$. It follows that $|I_v| \geq \frac{\ell}{3}$.

Thus, the probability that any particular vertex u is in $U_i^{'}$ for some particular i is at most $3/f^2\ell$. Thus, the expected number of neighbours of a given vertex

v in U_i' for any fixed i is at most $(2f\ell) \times (3/f^2\ell) = 6/f = \mathrm{o}(1)$. So for any particular i, U_i' is a stable set with very high probability. Furthermore, for any fixed v the following property holds with very high probability:

$$||N(v) \cap (\cup_{i=1}^{\ell} U_i')| - \tfrac{|N(v)\cap U|}{f^2}| \leq \sqrt{\log f\ell}\sqrt{\frac{|N(v) \cap U|}{f^2}}.$$

If desired, we could use the Local Lemma and insist that weakenings of these two properties simultaneously hold for all v and i with positive probability, thus obtaining:

Lemma 2 *Let G be a Δ-regular graph where $\Delta \geq \Delta_0 = \mathrm{e}^{10^7}$; and set $k = \lceil \log^2 \Delta \rceil$, $\ell = \lceil \frac{\Delta}{k} \rceil$, and $f = \lceil 3\sqrt{\log \Delta}\sqrt{k} \rceil$, and let $\{S_1, \ldots, S_\ell, U\}$ be a partition of $V(G)$ as assured by Lemma 1. Then there are disjoint stable sets $U_1', \ldots, U_\ell'$ in U such that*

1. *for each $1 \leq i \leq \ell - r$, $S_i + U_i'$ has a k colouring*
   and
   for each $\ell - r < i \leq \ell$, $S_i + U_i'$ has a $k-1$ colouring;
2. *for each $v \in V(G)$, $1 \leq i \leq \ell$, $|N(v) \cap U_i'| \leq \log \Delta$;*
3. *for each $v \in V(G)$,*

$$||N(v) \cap (\cup_{i=1}^{\ell} U_i')| - \tfrac{|N(v)\cap U|}{f^2}| \leq 3\sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap U|}{f^2}}.$$

We omit the proof of Lemma 2, as it is a special case of Lemma 3, to follow.

The first step of our iterative procedure consists essentially of applying Lemma 2 to obtain sets $U_1', \ldots, U_\ell'$ with the properties given in its statement. As we continue, we want to carry on using the Local Lemma in a similar manner. We have to be a little careful because as the number of elements of U decreases, so does $|U \cap N(v)|$ and hence our bound on $|I_v|$ for each vertex v. Furthermore, the ratio

$$\frac{\max_v |N(v) \cap U|}{\min_v |N(v) \cap U|}$$

begins to move away from 2 (recall that this ratio was important when we derived our bound on $|I_v|$). However, we will see that this does not pose a serious problem because as long as we halt our procedure before $|U \cap N(v)|$ drops below $f^{2.5}$, $|I_v|$ never gets too small, and this ratio remains bounded by $\sqrt{\log \Delta}$ which is enough for our purposes.

A problem which causes more difficulty is that of ensuring that each U_i' is a stable set. This is straightforward when a and b are large, but as they get smaller we have to use more care, and it is not enough to simply select these sets in the manner described in the discussion preceding Lemma 2.

We prove the following lemma.

Lemma 3 *Let Δ, k, ℓ, a and b be integers satisfying $\Delta \geq \Delta_0 = \mathrm{e}^{10^7}$, $k \geq \lceil \log^2 \Delta \rceil$, $\ell = \lceil \frac{\Delta}{k} \rceil$, $f = \lceil 3\sqrt{\log \Delta}\sqrt{k} \rceil$, and $f^{2.5} \leq a \leq b \leq a\sqrt{f}$. Let G be any Δ-regular graph and let $\{S_1, \ldots, S_\ell, U_1, \ldots, U_\ell, R\}$ be a partition of $V(G)$ which satisfies*

- *(a) for each $v \in V(G)$, $1 \leq i \leq \ell$, $k - 3f \leq |N(v) \cap S_i| \leq k - 2$;*
- *(b) for each $v \in V(G)$, $a \leq |N(v) \cap R| \leq b$;*
- *(c) for each $1 \leq i \leq \ell - r$, $S_i + U_i$ has a k colouring*

  and

  for each $\ell - r < i \leq \ell$, $S_i + U_i$ has a $k - 1$ colouring.

then there are disjoint stable sets $U_1^{'}, \ldots, U_\ell^{'}$ in R such that

1. *for each $1 \le i \le \ell - r$, $S_i + U_i + U_i^{'}$ has a k colouring*

   and

   for each $\ell - r < i \le \ell$, $S_i + U_i + U_i^{'}$ has a $k-1$ colouring;

2. *for each $v \in V(G)$, $1 \le i \le \ell$, $|N(v) \cap U_i^{'}| \le 4 \log^{1.5} \Delta$;*

3. *for each $v \in V(G)$,*

$$||N(v) \cap (\cup_{i=1}^{\ell} U_i^{'})| - \tfrac{|N(v) \cap R|}{f^2}| \le 4\sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap U|}{f^2}}.$$

Proof We prove this in two steps, each time using the symmetric form of the Local Lemma.

Claim 1: *There exists $U^{'} \subseteq R$ such that for all $v \in V(G)$,*

$$||N(v) \cap U^{'}| - \tfrac{|N(v) \cap R|}{f^2}| \le 3\sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap U|}{f^2}}.$$

Proof For each $u \in R$, place u in $U^{'}$ with probability f^{-2}, where this choice is made independently of the choice for all other vertices. For each $v \in V(G)$, let A_v be the event that $|N(v) \cap U^{'}|$ violates the guarantee of Claim 1. Note that by the Chernoff bounds, $\mathbf{Pr}(A_v) \le 2\Delta^{-3}$ and that each A_v is mutually disjoint of all but at most Δ^2 other events. Therefore, since $e \times 2\Delta^{-3} \times (\Delta^2 + 1) < 1$, the probability that $U^{'}$ meets the condition of Claim 1 is positive. $\square$

Most of $U^{'}$ will be placed in $\cup U_i^{'}$. In order to ensure that $S_i + U_i + U_i^{'}$ can be coloured as claimed, we will insist that for every $v \in U_i^{'}$, $|N(v) \cap (S_i \cup U_i)|$

is sufficiently small. To this end, for each $v \in U'$, we define $I_v = \{i : |N(v) \cap (S_i \cup U_i)| \leq k-1 \text{ and } |N(v) \cap (S_i \cup U_i)| \leq k-2 \text{ if } i > \ell - r\}$. Note that $|I_v| \geq a/(3f)$ by an argument similar to that preceding Lemma 2.

Claim 2: *There exists a partition $U'_1, \ldots, U'_\ell, X$ of U' such that*

1. *each U'_i is a stable set;*
2. *for all $v \in V(G)$, $1 \leq i \leq \ell$, if $v \in U'_i$ then $i \in I_v$;*
3. *for all $v \in V(G)$, $1 \leq i \leq \ell$, $|N(v) \cap U'_i| \leq 4 \log^{1.5} \Delta$;*
4. *for all $v \in V(G)$, $|N(v) \cap X| \leq \sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap R|}{f^2}}$.*

Note that this is enough to prove Lemma 3.

Claim 2 will follow from repeated iterations of:

Claim 3: *Given $Y \subseteq U'$, and sets $I'_v \subseteq I_v$ with $|I_v| \geq \frac{a}{6f}$ and $|N(v) \cap Y| < c$ for some $\sqrt{\log \Delta}\sqrt{\frac{a}{f^2}} \leq c \leq \frac{2b}{f^2}$, there exists a partition $Y_1, \ldots, Y_\ell, Y'$ of Y such that*

1. *each Y_i is a stable set;*
2. *for all $v \in Y$, $1 \leq i \leq \ell$, if $v \in Y_i$ then $i \in I'_v$;*
3. *for all $v \in V(G)$, $1 \leq i \leq \ell$, $|N(v) \cap Y_i| \leq 4\sqrt{\log \Delta}$;*
4. *for all $v \in V(G)$, $|N(v) \cap Y'| \leq \frac{c}{2}$.*

Proof: First we partition Y into $Y'_1, \ldots, Y'_\ell$ which satisfy conditions 2 and 3 of Claim 3. Next, for each i, we choose a stable set $Y_i \subseteq Y'_i$, such that condition 4 holds.

For each $v \in Y$, we select a uniformly random member $i \in I_v'$, making the choice independently of all other such choices, and put v into Y_i'.

Now, for each $1 \leq i \leq \ell$ and $v \in Y_i'$, if v has a neighbour in Y_i', then we put v into Y'. Finally we set $Y_i = Y_i' - Y'$.

For each $v \in V(G)$, let B_v be the event that v violates condition 3 for some i, and let C_v be the event that v violates condition 4. For each u, i, the probability that u is placed into Y_i' is at most $6f/a$, and so

$$\begin{aligned}
\mathbf{Exp}(|N(v) \cap Y_i|) &\leq \frac{6f}{a} \times c \\
&\leq \frac{6f}{a} \times \frac{b}{f^2} \\
&\leq \frac{6f}{a} \times \frac{2a\sqrt{f}}{f^2} \\
&\leq 12 f^{-1/2} < 1.
\end{aligned}$$

Therefore, by the Chernoff bounds,

$$\mathbf{Pr}(B_v) \leq \ell \times \mathrm{e}^{-(4\sqrt{\log \Delta})^2/3} < \Delta^{-4}.$$

Similarly, the probability that u is placed into Y' is at most $12f^{-1/2}$, and so $\mathbf{Exp}(|N(v) \cap Y'| \leq 12cf^{-1/2} < c/4$. Therefore,

$$\mathbf{Pr}(C_v) \leq \mathrm{e}^{-c/8} < \Delta^{-4}.$$

Furthermore, each event is independent of all but at most $2\Delta^3$ other events. Therefore, since $\mathrm{e} \times \Delta^{-4} \times (2\Delta^3 + 1) < 1$, the probability that conditions 3 and 4 hold is positive. Conditions 1 and 2 are enforced by the nature of our selection, and so the claim is true. $\square$

Now it only remains to prove Claim 2.

Proof of Claim 2: Set $Y_0 = U'$, $U_i' = \emptyset$ and repeatedly apply Claim 3 with $Y = Y_i$, and $I_v' = \{i \in I_v | U_i' \cap N(v) = \emptyset\}$, and set $Y_{i+1} = Y'$, $U_i' = U_i' \cup Y_i$.

Halt when $|N(v) \cap Y_i| \leq \sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap U'|}{f^2}}$, and set $X = Y_i$. Note that we will require fewer than $\log \Delta$ iterations, and so $|N(v) \cap U_i'| \leq 4 \log^{1.5} \Delta$. $\square$

Iterated applications of Lemma 3 now allow us to prove Theorem 1:

Proof of Theorem 1 First, we assume that G is Δ-regular, as if it is not, then G is a subgraph of some Δ-regular G' formed by adding edges and at most Δ vertices to G, such that it suffices to find such a colouring of G'.

We then begin by taking a partition of $V(G)$ as assured by Lemma 1, and we consider $\gamma = \lceil f^2(\log(f\ell) - \log(f^3))\rceil$ iterations of Lemma 3, each time replacing U_i by $U_i \cup U_i'$ from the previous iteration. During the jth iteration, we can take a, b to be a_j, b_j which are defined iteratively below.

Set $a_0 = f\ell, b_0 = 3f\ell$, $\alpha = 1 - 1/f^2$, and for $0 \leq j \leq \gamma - 1$ set

$$\begin{aligned} a_{j+1} &= a_j\left(1 - \frac{1}{f^2}\right) - 4\sqrt{\log \Delta}\frac{\sqrt{a_j}}{f} \\ &\geq \alpha a_j - \sqrt{a_j}, \end{aligned}$$

$$\begin{aligned} b_{j+1} &= b_j\left(1 - \frac{1}{f^2}\right) + 4\sqrt{\log \Delta}\frac{\sqrt{a_j}}{f} \\ &\leq \alpha b_j + \sqrt{b_j}. \end{aligned}$$

To ensure that we can carry out these iterations, we only need to bound b_j/a_j.

Note that $a_j \le \alpha^j a_0$, $b_j \ge \alpha^j b_0$, and both sequences are strictly decreasing.

To bound a_j, we set $j_0 = 0$, $j_{i+1} = j_i + \lceil \frac{1}{4} \log a_{j_i} / \log(\alpha^{-1}) \rceil$, and $\rho = (\frac{4}{3})^{1/4}$. Note that

$$\begin{aligned} a_{j_{i+1}} &\ge (a_{j_i})^{3/4} - (j_{i+1} - j_i)(a_{j_i})^{1/2} \\ &\ge \rho (a_{j_i})^{3/4} \\ &\ge \rho^{-(i+1)} \alpha^{j_{i+1}} a_0. \end{aligned}$$

Similarly, we set $t_0 = 0$, and $t_{i+1} = t_i + \lceil \frac{1}{4} \log b_{j_i} / \log(\alpha^{-1}) \rceil$. Note that

$$\begin{aligned} b_{t_{i+1}} &\le (b_{t_i})^{3/4} + (t_{i+1} - t_i)(b_{t_i})^{1/2} \\ &\le \rho (b_{t_i})^{3/4} \\ &\le \rho^{i+1} \alpha^{t_{i+1}} b_0. \end{aligned}$$

Set $i^* = \lceil \log\log \Delta / \log(\frac{4}{3}) \rceil$. For $j \le \gamma$, $j \le j_{i^*}, t_{i^*}$, and so $a_j \ge \alpha^j a_0 / \log^{1/4} \Delta \ge f^{2.5}$ and $b_j \le \log^{1/4} \Delta \alpha^j b_0 \le a_j \sqrt{f}$ as desired.

Therefore, we can partition $V(G)$ into sets $\{S_1, \dots, S_\ell, U_1, \dots, U_\ell, R\}$ such that

(a) for each $v \in V(G)$, $1 \le i \le \ell$, $k - 3f \le |N(v) \cap S_i| \le k - 2$;

(b) for each $v \in V(G)$, $|N(v) \cap R| \le f^3 \log^{1/4} \Delta$;

(c) for each $v \in V(G)$, $1 \le i \le \ell$, $|N(v) \cap U_i'| \le 4\gamma \log^{1.5} \Delta$;

(d) for each $1 \le i \le \ell - r$, $S_i + U_i$ has a k colouring

and

for each $\ell - r < i \le \ell$, $S_i + U_i$ has a $k - 1$ colouring.

Therefore, by colouring $V(G) - R$ with Δ colours as described in (d), and then greedily $\Delta + 1$ colouring R, using one more colour, we have a $\Delta + 1$ colouring of G in which for each $v \in V(G)$, $N(v)$ contains fewer than $k + 4\gamma \log^{1.5} \Delta + f^3 \log^{1/4} \Delta \leq \log^5 \Delta$ vertices from each colour class. $\square$

3 Smaller Values of β

Here, we consider the problem of finding the minimum number of colours necessary to produce a β-frugal colouring for general β. Our main results is the following;

Theorem 2 *For any $\beta \geq 1$ and sufficiently large Δ, every graph of maximum degree Δ has a β-frugal* $\max((\beta+1)\Delta, e^3\Delta^{1+1/\beta}/\beta)$ *colouring.*

Proof We use the general form of the Local Lemma. The case $\beta = 1$ is known, so we assume $\beta \geq 2$.

Let G be any graph with maximum degree Δ, and let $c = \max((\beta+1)\Delta, e^3\Delta^{1+1/\beta}/\beta)$. For each $v \in V(G)$, randomly choose one of c colours for v, where the choice is uniform over all colours, and independent of the choice made for any other vertex.

For each $(u, v) \in E(G)$, define $A_{u,v}$ to be the event that u, v are both in the same colour class. For each $(v, x_1, \ldots, x_{\beta+1})$ such that $\{x_1, \ldots, x_\beta\} \subseteq N(v)$ define $B_{v,x_1,\ldots,x_{\beta+1}}$ to be the event that $x_1, \ldots, x_{\beta+1}$ are all in the same colour class. All events $A_{u,v}$ are said to be Type A events and all events $B_{v,x_1,\ldots,x_{\beta+1}}$ are said to be Type B events.

$$\mathbf{Pr}(A_{u,v}) = \frac{1}{c},$$

and

$$\mathbf{Pr}(B_{v,x_1,\ldots,x_{\beta+1}}) = \frac{1}{c^\beta}.$$

Furthermore, each Type A event is independent of all but at most 2Δ Type A events and at most $2\Delta^{\beta+1}/b!$ Type B events. Similarly, each Type B event is independent of all but at most $(\beta+1)\Delta$ Type A events and at most $(\beta+1)\Delta^{\beta+1}/b!$ Type B events. Therefore, by setting $x_A = 1/((b+1)\Delta)$, $x_B = b!/((b+1)\Delta^{b+1})$, and applying the symmetric form of the local lemma, we see that the probability that none of these events occurs is positive, and thus there exists a β-frugal colouring. $\square$

We will now see that Theorem 2 is best possible up to a constant multiple:

Fact: *For any $\beta \geq 1$ and for arbitrarily high Δ, there is a graph of maximum degree Δ, which does not admit a β-frugal colouring using fewer than $\Delta^{1+1/\beta}/2\beta$ colours.*

Proof We are grateful to Noga Alon for the following example.

Consider a $(\beta+2)$-dimensional projective geometry, P, with $m = n^{\beta+1} + n^\beta + \ldots + 1$ points, every $(\beta+1)$-flat containing exactly $\Delta = n^\beta + \ldots + 1$ points, and n arbitrarily large. Form a bipartite graph G with parts P, F, where P is the set of points, and F is the set of $(\beta+1)$-flats, and where two vertices $p \in P$, $f \in F$ are adjacent if p lies in f.

Because P is a projective geometry, every set of $\beta+1$ points lies in a $(\beta+1)$-flat, and so in any β-frugal colouring of G, no colour can be used more than

β times. Therefore, at least m/β points are required. The result now follows as G has maximum degree Δ, and $m > \frac{1}{2}\Delta^{1+1/\beta}$. $\square$

Note that this also implies that Theorem 1 is nearly best possible.

Corollary 1 *For any $t > 1$, there exist graphs with maximum degree Δ arbitrarily large, which admit no $\log\Delta/\log\log\Delta$-frugal colouring using $t\Delta$ colours.*

Proof For $\beta = \log\Delta/\log\log\Delta$, $\Delta = \mathrm{o}(\Delta^{1+1/\beta}/\beta)$. $\square$

4 Algorithms

Beck [2] has developed a technique which makes some applications of the Local Lemma constructive (see also [1]). We will see in this section that this technique applies to Theorems 1 and 2, yielding efficient algorithms at the price of an increase of Δ_0 and the constant term in Theorem 2. We will need the following theorem, which we state in a somewhat more general form than is found in [2].

Theorem 3 *Let $\mathcal{A} = \{A_1, \ldots, A_r\}$ be a collection of subsets of a finite set X, $|X| = n$, each of size at most m such that each A_j intersects at most $m^{\epsilon^2/50}$ other members of $\mathcal{A}$ for some given $\epsilon > 60$. Suppose that we are given $\gamma_1, \gamma_2, \ldots \gamma_l$ with $\gamma_i = \gamma_i(n) > 1/m$, $\sum \gamma_i = 1$. Then we have an $\mathrm{O}(nrl \log^{\mathrm{O}(1)} n)$ time randomized algorithm and a polytime deterministic algorithm which finds a partition $X = X_1 \cup \ldots \cup X_l$ such that for each A_j,*

$$||X_i \cap A_j| - \gamma_i|A_j|| \leq \epsilon\sqrt{\gamma_i m \log m} \qquad 1 \leq i \leq l \qquad (*)$$

The case $l = 2, \gamma_1 = \gamma_2 = \frac{1}{2}$ is Theorem 5 of [2]. We omit the proof as it follows along the same lines as that in [2]. For ease of exposition, we present the randomised version of the algorithm. We start with the case where $m, l = \mathrm{O}(1)$.

Note that the *existence* of the desired partition is guaranteed by a straightforward application of the Local Lemma. The difficulty is in finding the partition efficiently.

The idea is that if we place each $x \in X$ into a randomly chosen part where each X_i is chosen with probability γ_i, then with high probability (*) holds for most of the subsets A_j. In order to ensure that (*) holds for *every* A_j, we make these choices one-at-a-time, and if during this process any A_y comes too close to violating (*) we stop choosing parts for all remaining members of A_j, postponing these choices until a later time when we will be more careful.

Step 1: Arbitrarily order the elements of X, $x_1, \ldots, x_n$. Set $SKIP = \emptyset$. For $k = 1, \ldots, n$, do the following:

(a) If $x_k \in SKIP$ then next k. Otherwise, randomly choose which part to place x_k into, placing it into X_j with probability γ_j.

(b) For each $1 \leq i \leq l, 1 \leq j \leq r$, if $||A_j \cap X_i| - \gamma_i |A_j \cap \cup_{i'=1}^{l} X_{i'}|| \geq \frac{1}{3}\epsilon\sqrt{\gamma_i m \log m}$ then set $SKIP = SKIP \cup (A_j - \cup_{i'=1}^{l} X_{i'})$, thus postponing the choice in step 1(a) for the rest of the members of A_j.

At this point all of $X - SKIP$ has been distributed into $X_1, ..., X_\ell$, and all that remains is to distribute the members of SKIP. It can be shown using the first moment method that with high probability each component of the hypergraph with vertex set SKIP, and edges $A_j \cap SKIP$ has at most $\mathrm{O}(\log r)$ edges. We would like to reduce these component sizes even further, so we repeat step 1 one more time.

Step 2: Set $X' = SKIP$ and for each $1 \leq j \leq r$, set $A_j' = A_j \cap SKIP$. Repeat step 1 using X' and $\mathcal{A}' = \{A_1', ..., A_r'\}$ rather that $\mathcal{A}$ and X.

Denote by $\mathcal{H}$ the hypergraph with vertex set SKIP and edges $A_j'' = A_j \cap SKIP$. With high probability each component of $\mathcal{H}$ has at most $\mathrm{O}(\log \log r)$ edges.

A straightforward application of the Local Lemma shows that there exists a partition $SKIP = Z_1 \cup ... \cup Z_l$ such that for each A_j'',

$$||Z_i \cap A_j| - \gamma_i |A_j|| \leq \frac{1}{3}\epsilon\sqrt{\gamma_i m \log m} \qquad 1 \leq i \leq l.$$

Clearly setting each $X_i = X_i \cup Z_i$ will yield the desired partition of X.

The key point of this algorithm is to note that we can look for the partition of the members of any component of $\mathcal{H}$ independently of our search for the partition of the rest of $SKIP$. Furthermore, the sizes of these components are small enough that we can find their partitions using exhaustive search! The number of possible partitions of a component is at most $l^{\mathrm{O}(m \log \log r)}$ and so if $m, l = \mathrm{O}(1)$ then we can check them all in time $\mathrm{O}(\log r)$:

Step 3: By exhaustive search, find a partition $SKIP = Z_1 \cup \ldots \cup Z_l$, where for $1 \leq i \leq l$ and $1 \leq j \leq r$, $||A_j'' \cap Z_i| - \gamma_i |A_j''|| \leq \frac{1}{3}\epsilon\sqrt{\gamma_i m \log m}$.

If l grows with r, then we must first split $SKIP$ into 2 parts, and then split each part into 2 more, and so on until we have our ℓ parts.

Step 3': By exhaustive search, find a partition $SKIP = Y_1 \cup Y_2$ such that, with $\delta_1 = \prod_{i=1}^{\lceil \frac{l}{2} \rceil} \gamma_i, \delta_2 = \prod_{i=\lceil \frac{l}{2} \rceil+1}^{r} \gamma_i$, for $i = 1, 2$, and for $1 \leq j \leq r$, $||A_j'' \cap Y_i| - \delta_i |A_j''|| \leq \frac{1}{6}\epsilon\sqrt{\delta_i |A_j''| \log m}$. In the same manner, repeatedly split each part, until we have $SKIP = Z_1 \cup \ldots \cup Z_l$, where for $1 \leq i \leq l$ and $1 \leq j \leq r$, $||A_j'' \cap Z_i| - \gamma_i |A_j''|| \leq \frac{1}{3}\epsilon\sqrt{\gamma_i m \log m}$.

If m grows with r then we proceed as follows: If $m = \mathrm{O}(\log\log r)$ then we can still carry out each iteration of step 3' in poly$(\log r)$ time. Otherwise, with high probability, a random partition of each component of $\mathcal{H}$ will satisfy our desired property.

Using standard techniques for derandomizing the first moment method, such as those in [2], we can derandomise step 1, yielding a polytime deterministic algorithm.

Now, we will use Theorem 3 to find an efficient algorithm to produce a colouring as guaranteed by Theorem 1, with a slight increase of Δ_0.

The first step is to find a partition of $V(G)$ into $S_1, S_2, \ldots, S_\ell, U$ as in Lemma 1. Here, we increase f to $200\lceil\sqrt{\log \Delta}\sqrt{k}\rceil$. This is a straightforward application of Theorem 3 with $X = V(G)$, $A_v = N(v), v \in V(G)$, $l = \ell$ and $\gamma_i = (1 - \frac{3}{2}f\ell/\Delta)/\ell$.

The next step is to find stable sets $U_1', \ldots, U_\ell' \subseteq R$ as described in Lemma 3. We weaken conditions 2,3 of Lemma 3 to:

1. for each $v \in V(G)$, $1 \leq i \leq \ell$, $|N(v) \cap U_i'| \leq 600 \log^{1.5} \Delta$;

2. for each $v \in V(G)$,

$$||N(v) \cap (\cup_{i=1}^{\ell} U_i')| - \tfrac{|N(v) \cap R|}{f^2}| \leq 600\sqrt{\log \Delta}\sqrt{\frac{|N(v) \cap U|}{f^2}}.$$

We can find U' as in Claim 1 by Theorem 3, with $X = R$, $A_v = N(v) \cap R$, $l = 1$ and $\gamma_1 = f^{-2}$. To choose U_i', we use an algorithm similar to that given by Theorem 3:

Step 1: Arbitrarily order the vertices of U', $u_1, \ldots, u_t$. Set $SKIP = \emptyset$. For $k = 1, \ldots, t$, do the following:

(a) If $u_k \in SKIP$ then next k. Otherwise, place u_k into U_{i^*}', where i^* is chosen uniformly at random from I_u. For each $s \in N(u) \cap U'$, set $I_s = I_s - \{i^*\}$.

(b) For each i, v, if $|N(v) \cap U_i'| \geq 100 \log^{1.5} \Delta$, then set $SKIP = SKIP \cup (N(v) - \cup_{i'=1}^{\ell} U_{i'}')$.

Step 2: Repeat step 1 with $U' - \cup_{i=1}^{\ell} U_i'$ rather than U.

Step 3: Now, by exhaustive search, choose the remainder of $U_i{}'$ in the same manner as in the previous algorithm.

The proof that this algorithm succeeds with high probability follows along the same lines as the proof of Theorem 3 and we omit the details. Again, this can be derandomized, providing a polytime deterministic algorithm.

We must apply Lemma 3 $\gamma = \mathrm{O}(\log^4 \Delta)$ times, and then we simply greedily colour the graph, as described in the proof of Theorem 1, yielding a $\mathrm{O}(n^3 \log^{\mathrm{O}(1)} \Delta)$ time randomized algorithm, or a polytime deterministic algorithm.

5 Remarks

It is worth noting that by being more careful with our calculations, and raising Δ_0, we can find a $\log^3 \Delta$-frugal $(\Delta+1)$-colouring for any graph with maximum degree $\Delta \geq \Delta_0$. However, it seems that our methods will not be enough to find a $\log^c \Delta$-frugal $(\Delta+1)$-colouring for c close to one.

On the other hand, by being slightly less frugal, we can decrease our rather large lower bound on Δ, showing, for example, that any graph with maximum degree $\Delta > 1$ has a $\log^{20} \Delta$-frugal $(\Delta+1)$-colouring.

Alon [1] has shown how to modify the technique of [2] to produce parallel algorithms. His methods do not seem to apply here.

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