A Gap Between the Appearances of a k-core and a $(k+1)$-chromatic Graph.

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Abstract

We observe that the values of p for which with high probability $G_{n,p}$ is k-colourable and for which with high probability $G_{n,p}$ has no k-core are not equal for $k \geq 4$.

Consider the random graph $G_{n,p}$ with n vertices and where each edge appears independently with probability p. Here, we are concerned with the value of

$$d_k = \sup\{d|\text{ w.h.p.}^1 \chi(G_{n,p=d/n}) \leq k\},$$

where $\chi(G)$ denotes the chromatic number of G. If $\chi(G) > k$ then G must have a subgraph with minimum degree at least k. We refer to the unique maximal such subgraph as the k-core of G, and define

$$c_k = \sup\{c|\text{ w.h.p. } G_{n,p=c/n} \text{ has no } k\text{-core}\}$$

[1] We say that $G_{n,p}$ has a property P *with high probability* (w.h.p.) if $\lim_{n\to\infty} \mathbf{Pr}\{G_{n,p} \text{ has } P\} = 1$.

Bollobás has asked if $d_k = c_k$ for all $k \geq 3$. (It is well-known that $d_2 = 0$ and $c_2 = 0$.)

The following relation is implicit in Bollobás and Thomason [1] (see also [2])

$$d_{k+1} \geq d_k + \log(d_{k+1}), \qquad k \geq 2. \tag{1}$$

They prove (1) by showing that w.h.p. a greedy independent set algorithm applied to $G_{n,p=d/n}$ will yield an independent set of size $z = \zeta n + \mathrm{o}(n)$, where $\zeta = \log d/d$. Furthermore, the graph formed by deleting this stable set is identical in distribution to $G_{n',p}$ where $n' = n - z = (1-\zeta)n + \mathrm{o}(n)$, and $p = d/n = d(1-\zeta)/n'$. Therefore, if $d(1-\zeta) < d_k$, then w.h.p. $G_{n,p=d/n}$ is $(k+1)$-colourable.

Devroye (see [2]) has shown $d_k \leq 2\log k/(\log k - \log(k-1))$, and Łuczak [4] showed that in fact $d_k = 2k\log k + \mathrm{o}(k)$. Łuczak [3] also showed that $c_k = k + \mathrm{O}(k^{1/2+\epsilon})$ for any $\epsilon > 0$, thus establishing that $c_k < d_k$ for large k.

Recently Pittel, Spencer and Wormald [5] have determined

$$\begin{aligned} c_k &= \min_{\lambda>0} \lambda/\pi_k(\lambda) \\ &= k + \sqrt{k\log k} + \mathrm{O}(\log k), \end{aligned}$$

where $\pi_k(\lambda) = \mathbf{Pr}\{\mathrm{Poisson}(\lambda) \geq k-1\}$. Here, we see that this can be used to establish:

Theorem 1 $c_k < d_k$ *for* $k \geq 4$.

Proof: From $d_3 \geq c_3 = 3.35\ldots$ and (1), we have $d_4 \geq 5.16 > c_4 = 5.14\ldots$. Repeated iterations of (1) yield $d_k > c_k$ for $4 \leq k \leq 40$.

For $k > 40$ it is easily verified by setting $\lambda = 2k$ that $c_k \leq 2.3k$. On the other hand, a straightforward induction shows that $d_k > k \log k > 3k$ for $k \geq 40$. $\square$

This gap grows quickly - for example, $c_{40} = 52.23\ldots$ while $d_{19} > 53.88$. Thus for $p = 53/n$ w.h.p. $G_{n,p}$ has a subgraph with minimum degree 40, but is 19-colourable.

The relation between c_3 and d_3 is still unknown. However, in light of Theorem 1, it seems unlikely that $c_3 = d_3$.

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References

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