The scaling window for a random graph with a given degree sequence

Hamed Hatami and Michael Molloy*

Department of Computer Science

University of Toronto

e-mail: hamed@cs.toronto.edu, molloy@cs.toronto.edu

Abstract

We consider a random graph on a given degree sequence $\mathcal{D}$, satisfying certain conditions. We focus on two parameters $Q = Q(\mathcal{D}), R = R(\mathcal{D})$. Molloy and Reed proved that $Q = 0$ is the threshold for the random graph to have a giant component. We prove that if $|Q| = O(n^{-1/3}R^{2/3})$ then, with high probability, the size of the largest component of the random graph will be of order $\Theta(n^{2/3}R^{-1/3})$. If Q is asymptotically larger/smaller that $n^{-1/3}R^{2/3}$ then the size of the largest component is asymptotically larger/smaller than $n^{2/3}R^{-1/3}$. In other words, we establish that $|Q| = O(n^{-1/3}R^{2/3})$ is the scaling window.

1 Introduction

The double-jump threshold, discovered by Erdős and Rényi[17], is one of the most fundamental phenomena in the theory of random graphs. The component structure of the random graph $G_{n,p=c/n}$ changes suddenly when c moves from below one to above one. For every constant $c < 1$, almost surely[1] (a.s.) every component has size $O(\log n)$, at $c = 1$ a.s. the largest component has size of order $\Theta(n^{2/3})$, and at $c > 1$ a.s. there exists a single giant component of size $\Theta(n)$ and all other components have size $O(\log n)$.

Bollobás[8], Łuczak[28] and Łuczak et al.[29] studied the case where $p = \frac{1+o(1)}{n}$. Those papers showed that when $p = \frac{1}{n} \pm O(n^{-1/3})$, the component sizes of $G_{n,p}$ behave as described above for $p = \frac{1}{n}$[29]. Furthermore, if p lies outside of that range, then the size of the largest component behaves very differently: For larger/smaller values of p, a.s. the largest component has size asymptotically larger/smaller than $\Theta(n^{2/3})$[8, 28]. That range of p is referred to as the *scaling window*. See eg. [9] for further details.

*Research supported by an NSERC Discovery Grant.

[1] A property P holds *almost surely* if $\lim_{n\to\infty} \mathbf{Pr}(P) = 1$.

This is a classical example of one of the leading thrusts in the study of random combinatorial structures: to determine thresholds and analyze their scaling windows. Roughly speaking, a threshold is a point at which the random structure changes dramatically in a certain sense, and a study of the scaling window examines the structure as it is undergoing that change. For example, the threshold for random 2-SAT has been known since the early 90's [14, 16, 19]; Bollobás et al[10] determined that it also has a scaling window of width $O(n^{-1/3})$. Besides mathematicians, these problems are also pursued by statistical physicists, who model the sudden transitions undergone by various physical systems. See [12, 3, 20, 26, 31] for just a few other examples of such studies.

Molloy and Reed[32] proved that something analogous to the cases $c < 1$ and $c > 1$ of the Erdős-Rényi double-jump threshold holds for random graphs on a given degree sequence. They considered a sequence $\mathcal{D} = (d_1, ..., d_n)$ satisfying certain conditions, and chose a graph uniformly at random from amongst all graphs with that degree sequence. They determined a parameter $Q = Q(\mathcal{D})$ such that if $Q < 0$ then a.s. every component has size $O(n^x)$ for some $x < 1$ and if $Q > 0$ then a.s. there exists a giant component of size $\Theta(n)$ and all others have size $O(\log n)$.

Aiello, Chung and Lu[1] applied the results of Molloy and Reed[32, 33] to analyze the connectivity structure of a model for massive networks. Those results have since been used numerous times for other massive network models arising in a wide variety of fields such as physics, sociology and biology (see eg. [36]).

The threshold $Q = 0$ was discovered more than a decade ago. Yet despite the substantial multidisciplinary interest, there has been no progress on what happens inside the scaling window. The work of Kang and Seierstad[24] and Jansen and

Łuczak[23] established bounds on the width of the scaling window, under certain conditions on $\mathcal{D}$: the largest component has size $\ll n^{2/3}$ for $Q \ll -n^{-1/3}$ and $\gg n^{2/3}$ for $Q \gg n^{-1/3}$ (see Section 1.2). These, of course, are results that one would aim for if motivated by the hypothesis that the scaling window for these degree sequences behaves like that of $G_{n,p}$. However, it was not known whether $|Q| = O(n^{-1/3})$ really is the scaling window we should be aiming for, nor whether $\Theta(n^{2/3})$ is the component size that we should be aiming for.

In this paper, we establish the scaling window and the size of the largest component when Q is inside the scaling window, under conditions for $\mathcal{D}$ that are less restrictive than the conditions from[32, 33, 24, 23]. We will state our results more formally in the next subsection, but in short: If $\sum d_i^3 = O(n)$, then the situation is indeed very much like that for $G_{n,p}$. The scaling window is the range $Q = O(n^{-1/3})$ and inside the scaling window, the size of the largest component is $O(n^{2/3})$. As discussed below, the conditions required in [24, 23] imply that $\sum d_i^3 = O(n)$, which explains why they obtained those results. If $\sum d_i^3 \gg n$, then the situation changes: the size of the scaling window becomes asympotically larger, and the size of the largest component becomes asymptotically smaller.

1.1 The main results We are given a set of vertices along with the degree d_v of each vertex. We denote this degree sequence by $\mathcal{D}$. Our random graph is selected uniformly from amongst all graphs with degree sequence $\mathcal{D}$. We assume at least one such graph exists and so, eg., $\sum_v d_v$ is even. We use $\mathcal{C}_{\max}$ to denote the largest component of this random graph.

We use E to denote the set of edges, and note that $|E| = \frac{1}{2}\sum_{v\in G} d_v$. We let n_i denote the number of vertices of degree i. We define:

$$Q := Q(\mathcal{D}) := \frac{\sum_{u\in G} d_u^2}{2|E|} - 2,$$

and

$$R := R(\mathcal{D}) := \frac{\sum_{u\in G} d_u(d_u-2)^2}{2|E|}.$$

The relevance of Q, R will be made clear in Section 2.3. The asymptotic order of R will be important; in our setting, $|E|/n$ and Q are bounded by constants, and so R has the same order as $\frac{1}{n}\sum_{u\in G} d_u^3$. The order of R was implicitly seen to be important in the related papers [23, 24], where they required $\frac{1}{n}\sum_{u\in G} d_u^3$ to be bounded by a constant (see Section 1.2).

Molloy and Reed [32] proved that, under certain assumptions about $\mathcal{D}$, if Q is at least a positive constant, then a.s. $|\mathcal{C}_{\max}| \geq cn$ for some $c > 0$ and if Q is at most a negative constant then a.s. $|\mathcal{C}_{\max}| \leq n^x$ for some constant $x < 1$. One assumption was that the degree sequence was *well-behaved* in that it converged in some sense as $n \to \infty$. We don't require that assumption here.

But we do require some assumptions about our degree sequence. First, it will be convenient to assume that every vertex has degree at least one. A random graph with degree sequence $d_1, ..., d_n$ where $d_i = 0$ for every $i > n'$ has the same distribution as a random graph with degree sequence $d_1, ..., d_{n'}$ with $n - n'$ vertices of degree zero added to it. So it is straightforward to apply our results to degree sequences with vertices of degree zero.

The case $n_2 = n - o(n)$ forms an anomaly. For example, in the extreme case where $n_2 = n$, we have a random 2-regular graph, and in this case the largest component is known to have size $\Theta(n)$ (see eg. [2]). So we require that $n_2 \leq (1-\zeta)n$ for some constant $\zeta > 0$, as did [23, 24]. See Remark 2.7 of [23] for a description of other behaviours that can arise when we allow $n_2 = n - o(n)$.

As in [32, 33] and most related papers (eg. [18, 23, 24]), we require an upper bound on the maximum degree, Δ. We take $\Delta \leq n^{1/3}R^{1/3}(\ln n)^{-1}$ (which is higher than the bounds from [23, 24, 32, 33] and comparable to that from [18]).

Finally, since we are concerned with $Q = o(1)$, we can assume $|Q| \leq \frac{\zeta}{2}$, and that ζ is sufficiently small, eg. $\zeta < \frac{1}{10}$. In summary, we assume that $\mathcal{D}$ satisfies the following:

Condition D: *For some constant* $0 < \zeta < \frac{1}{10}$*: (a)* $\Delta \leq n^{1/3}R^{1/3}(\ln n)^{-1}$*; (b)* $n_0 = 0$*; (c)* $n_2 \leq (1-\zeta)n$*; (d)* $|Q| \leq \frac{\zeta}{2}$.

Our main theorems are:

THEOREM 1.1. *For any* $\lambda, \epsilon, \zeta > 0$ *there exist* A, B *and* N *such that for any* $n \geq N$ *and any degree sequence* $\mathcal{D}$ *satisfying Condition D and with* $-\lambda n^{-1/3}R^{2/3} \leq Q \leq \lambda n^{-1/3}R^{2/3}$*, we have*

(a) $\Pr[|\mathcal{C}_{\max}| \leq An^{2/3}R^{-1/3}] \leq \epsilon$*;*

(b) $\Pr[|\mathcal{C}_{\max}| \geq Bn^{2/3}R^{-1/3}] \leq \epsilon$.

THEOREM 1.2. *For any* $\epsilon, \zeta > 0$ *and any function* $\omega(n) \to \infty$*, there exists* B, N *such that for any* $n \geq N$ *and any degree sequence* $\mathcal{D}$ *satisfying Condition D and with* $Q < -\omega(n)n^{-1/3}R^{2/3}$*:*

(a) $\mathbf{Pr}(|\mathcal{C}_{\max}| \geq B\sqrt{n/|Q|}) < \epsilon$.

(b) $\mathbf{Pr}(\exists$*a component with more than one cycle*$) < \frac{20}{\omega(n)^3}$.

THEOREM 1.3. *For any $\epsilon, \zeta > 0$ and any function $\omega(n) \to \infty$, there exists A, N such that for any $n \geq N$ and any degree sequence $\mathcal{D}$ satisfying Condition D and with $Q > \omega(n)n^{-1/3}R^{2/3}$:*

$$\mathbf{Pr}(|\mathcal{C}_{\max}| \leq AQn/R) < \epsilon.$$

Note that the bounds on $|\mathcal{C}_{\max}|$ in Theorems 1.2 and 1.3 are $B\sqrt{n/|Q|} < Bn^{-1/3}R^{2/3}/\sqrt{\omega(n)}$ and $AQn/R > A\omega(n)n^{2/3}R^{-1/3}$. So our theorems imply that $|Q| = O(n^{-1/3}R^{2/3})$ is the scaling window for any degree sequences that satisfy Condition D, and that in the scaling window the size of the largest component is a.s. $\Theta(n^{2/3}R^{-1/3})$.

Note also that Theorem 1.2(b) establishes that when Q is below the scaling window, then a.s. every component is either a tree or is unicyclic. This was previously known to be the case for the $G_{n,p}$ model[28].

The approach we take for Theorems 1.1 and 1.3 closely follows that of Nachmias and Peres[34] who applied some Martingale analysis, including the Optional Stopping Theorem, to obtain a short elegant proof of what happens inside the scaling window for $G_{n,p=c/n}$. See also [35] where they apply similar analysis to also obtain a short proof of what happens outside the scaling window, including tight bounds on the size of the largest component.

1.2 Related Work Van der Hofstad[21] obtained similar results on the scaling windows for models of inhomogeneous random graphs in which the expected degree sequence exhibits a power law. (An inhomogeneous random graph is one in which the edges between pairs of vertices are chosen independently, but with varying probabilities.) A critical point for such graphs was determined by Bollobás et al[11]. Van der Hofstad showed that if the exponent τ of the power law is at least 4, then the size of the scaling window has size at least $n^{-1/3}$, and in that window, the size of the largest component is $\Theta(n^{2/3})$; when $3 < \tau < 4$, those values change to $n^{-(\tau-3)/(\tau-1)}$ and $\Theta(n^{(\tau-2)/(\tau-1)})$. ($Q = O(1)$ implies $\tau > 3$.) In that setting, $\tau \geq 4$ corresponds to $R = O(1)$. These sizes are equal to the corresponding values from Theorem 1.1, although in the case $3 < \tau < 4$, the expected value of Δ satisfies $\Delta = O(n^{1/3}R^{1/3})$ and so the expected degree sequence would not satisfy Condition D. See also [5, 6] for more detailed results.

Cooper and Frieze[15] proved, amongst other things, an analogue of the main results of Molloy and Reed[32, 33] in the setting of giant strongly connected components in random digraphs.

Fountoulakis and Reed[18] extended the work of [32] to degree sequences that do not satisfy the convergence conditions required by [32]. They require $\Delta \leq |E|^{1/2-\epsilon}$ which in their setting implies $\Delta \leq O(n^{1/2-\epsilon})$.

Kang and Seierstad[24] applied generating functions to study the case where $Q = o(1)$, but is outside of the scaling window. They require a maximum degree of at most $n^{1/4-\epsilon}$ and that the degree sequences satisfy certain conditions that are stronger than those in [32]; one of these conditions implies that R is bounded by a constant. They determine the (a.s. asymptotic) size of $|\mathcal{C}_{\max}|$ when $Q \ll -n^{-1/3}$ or $Q \gg n^{-1/3}\log n$. In the former, $|\mathcal{C}_{\max}| \ll n^{2/3}$ and in the latter, $|\mathcal{C}_{\max}| \gg n^{2/3}$. So for the case where $R = O(1)$ is bounded, this almost showed that the scaling window is not larger than the natural guess of $Q = O(n^{2/3})$ - except that it left open the range where $n^{-1/3} \ll Q = O(n^{-1/3}\log n)$.

Jansen and Luczak[23] use simpler techniques to obtain a result along the lines of that in [24]. They require a maximum degree of $n^{1/4}$, and they also require $R = O(1)$; in fact, they require $\frac{1}{n}\sum_v d_v^{4+\eta}$ to be bounded by a constant (for some arbitrarily small constant $\eta > 0$), but they conjecture that having $\frac{1}{n}\sum_v d_v^3$ bounded (i.e. R bounded) would suffice. For $Q \gg n^{-1/3}$, they determine $|\mathcal{C}_{\max}|$, and show that it is a.s. asymptotically larger than $n^{2/3}$. Thus (in the case that their conditions hold) they eliminated the gap left over from [24]. They also use their techniques to obtain a simpler proof of the main results from [32, 33].

So for the case $R = O(1)$, Theorems 1.2(a) and 1.3 were previously known (under somewhat stronger conditions). But there was nothing known about when Q is inside the scaling window. In fact, it was not even known that $Q = O(n^{-1/3})$ was the scaling window; it was possibly smaller. And nothing was known for the case when R grows with n.

2 Preliminaries

2.1 The Random Model In order to generate a random graph with a given degree sequence $\mathcal{D}$, we use the *configuration model* due to Bollobás[7] and inspired by Bender and Canfield[4]. In particular, we: (1) Form a set L which contains d_v distinct copies of every vertex v. (2) Choose a random perfect matching over the elements of L. (3) Contract the different copies of each vertex v in L into a single vertex.

This may result in a multigraph, but a standard argument yields:

PROPOSITION 2.1. *Consider any degree sequence $\mathcal{D}$ satisfying Condition D. Suppose that a property $\mathcal{P}$ holds with probability at most ϵ for a uniformly random configuration with degree sequence $\mathcal{D}$. Then for a uniformly random graph with degree sequence $\mathcal{D}$, $\mathbf{Pr}(\mathcal{P}) \leq \epsilon \times e$.*

2.2 Martingales A random sequence $X_0, X_1, ...$ is a *martingale* if for all $i \geq 0$, $\mathbb{E}(X_{i+1}|X_0, ..., X_i) = X_i$. It is a *submartingale*, resp. *supermartingale*, if for all $i \geq 0$, $\mathbb{E}(X_{i+1}|X_0, ..., X_i) \geq X_i$, resp. $\leq X_i$.

A *stopping time* for a random sequence $X_0, X_1, ...$ is a step τ (possibly $\tau = \infty$) such that we can determine whether $i = \tau$ by examining only $X_0, ..., X_i$. It is often useful to view a sequence as, in some sense, halting at time τ; a convenient way to do so is to consider the sequence $X_{\min(i,\tau)}$, whose ith term is X_i if $i \leq \tau$ and X_τ otherwise.

In our paper, we will make heavy use of the Optional Stopping Theorem. The version that we will use is the following, which is implied by Theorem 17.6 of [27]:

The Optional Stopping Theorem *Let $X_0, X_1, ...$ be a martingale (resp. submartingale, supermartingale), and let $\tau \geq 0$ be a stopping time. If there is a fixed bound T such that $\mathbf{Pr}(\tau \leq T) = 1$ then $\mathbb{E}(X_\tau) = X_0$ (resp. $\mathbb{E}(X_\tau) \geq X_0$, $\mathbb{E}(X_\tau) \leq X_0$).*

2.3 The Branching Process As in [32], we will examine our random graph using a branching process of the type first applied to random graphs by Karp in [25]. Given a vertex v, we explore the configuration starting from v in the following manner: At step t, we will have a partial subgraph C_t which has been exposed so far. We will use Y_t to denote the total number of unmatched vertex-copies of vertices in C_t. So $Y_t = 0$ indicates that we have exposed an entire component and are about to start a new one.

1. Choose an arbitrary vertex v and initialize $C_0 = \{v\}$; $Y_0 = \deg(v)$.
2. Repeat while there are any vertices not in C_t:
   (a) If $Y_t = 0$, then pick a uniformly random vertex-copy from amongst all unmatched vertex-copies; let u denote the vertex of which it is a copy. $C_{t+1} := C_t \cup \{u\}$; $Y_{t+1} := \deg(u)$.
   (b) Else choose an arbitrary unmatched vertex-copy of any vertex $v \in C_t$. Pick as its partner a uniformly random vertex-copy from amongst all other unmatched vertex-copies; let u denote the vertex of which it is a copy. Thus we expose an edge uv.
      i. If $u \notin C_t$ then $C_{t+1} := C_t \cup \{u\}$; $Y_{t+1} := Y_t + \deg(u) - 2$.
      ii. Else $C_{t+1} := C_t$; $Y_{t+1} := Y_t - 2$.

For $t \geq 1$ let

- $\eta_t := Y_t - Y_{t-1}$.
- $D_t := Y_t + \sum_{u \notin C_t} d_u$, the total number of unmatched vertex-copies remaining at time t.
- $v_t := \emptyset$ if C_{t-1} and C_t have the same vertex set; else v_t is the unique vertex in $C_t \setminus C_{t-1}$.
- $Q_t := \frac{\sum_{u \notin C_t} d_u^2}{D_t - 1} - 2$.
- $R_t := \frac{4(Y_t - 1) + \sum_{u \notin C_t} d_u(d_u - 2)^2}{D_t - 1}$.

Note that Q_t and R_t begin at $Q_0 \approx Q$ and $R_0 \approx R$. Furthermore, for $u \notin C_t$, $\Pr[v_{t+1} = u] = \frac{d_u}{D_t - 1}$, and so if $Y_t > 0$ then the expected change in Y_t is

$$\begin{aligned} \mathbb{E}[\eta_{t+1}|C_t] &= (\sum_{u \notin C_t} \Pr[v_{t+1} = u] \times d_u) - 2 \\ &= \frac{\sum_{u \notin C_t} d_u^2}{D_t - 1} - 2 = Q_t. \qquad (2.1) \end{aligned}$$

If Q_t remains approximately Q, then Y_t is a random walk with drift approximately Q. So if $Q < 0$ then we expect Y_t to keep returning to zero quickly, and hence we only discover small components. But if $Q > 0$ then we expect Y_t to grow large; i.e. we expect to discover a large component. This is the intuition behind the main result of [32].

The parameter R_t measures the expected value of the square of the change in Y_t, if $Y_t > 0$:

$$\begin{aligned} \mathbb{E}[\eta_{t+1}^2|C_t] &= \Pr[v_{t+1} = \emptyset] \times 4 \\ &\quad + \sum_{u \notin C_t} \Pr[v_{t+1} = u] \times (d_u - 2)^2 \\ &= \frac{4(Y_t - 1) + \sum_{u \notin C_t} d_u(d_u - 2)^2}{D_t - 1} \\ &= R_t. \qquad (2.2) \end{aligned}$$

If $Y_t = 0$, then the expected values of η_{t+1} and η_{t+1}^2 are not equal to Q_t, R_t, as in this case:

$$\mathbb{E}[\eta_{t+1}|C_t] = \frac{\sum_{u \notin C_t} d_u^2}{D_t}, \qquad (2.3)$$

(2.4) $\mathbb{E}[\eta_{t+1}^2|C_t] = \dfrac{\sum_{u\notin C_t} d_u^3}{D_t} \geq R_t \times \dfrac{D_t-1}{D_t} \geq \dfrac{R_t}{2}.$

Note that, for $Y_t > 0$, the expected change in Q_t is approximately:

$$\begin{aligned}\mathbb{E}[Q_{t+1} - Q_t|C_t] &\approx -\sum_{u\notin C_t} \Pr[v_{t+1} = u] \times \frac{d_u^2}{D_t - 1}\\ &= -\frac{\sum_{u\notin C_t} d_u^3}{(D_t-1)^2}\end{aligned}$$

which, as long as $D_t = n - o(n)$, is asymptotically of the same order as $-\frac{R_t}{n}$. So if R_t remains approximately R, then Q_t will have a drift of roughly $-\frac{R}{n}$; i.e. the branching factor will decrease at approximately that rate. So amongst degree sequences with the same value of Q, we should expect those with large R to have $|\mathcal{C}_{\max}|$ smaller. This explains why $|\mathcal{C}_{\max}|$ is a function of both Q and R in Theorem 1.1.

The proofs of the following concentration bounds on Q_t, R_t appear in the full version of the paper.

Lemma 2.1. *For each* $1 \leq t \leq \frac{\zeta}{400}\frac{n}{\Delta}$,

$$\Pr[|R_t - R| \geq R/2] < n^{-10}.$$

Lemma 2.2. *For each* $1 \leq t \leq \frac{\zeta}{1000}\frac{|Q|n}{R} + 2n^{2/3}R^{-1/3}$,

$$\Pr\left[|Q_t - Q| > \frac{1}{2}|Q| + \frac{800}{\zeta}n^{-1/3}R^{2/3}\right] \leq n^{-10}.$$

3 Proof of Theorem 1.2

Proof of Theorem 1.2(a). The proof is somewhat along the lines of that of Theorem 1.1(b), but is much simpler since Lemma 2.2 allows us to assume that the drift Q_t is negative for every relevant t. The details appear in the full version of the paper. □

Proof of Theorem 1.2(b) As noted by Karonski for the proof of the very similar Lemma 1(iii) of [28]: if a component contains at least two cycles then it must contain at least one of the following two subgraphs:

- W_1 - two vertices u, v that are joined by three paths, where the paths are vertex-disjoint except for at their endpoints.
- W_2 - two edge-disjoint cycles, one containing u and the other containing v, and a (u, v)-path that is edge-disjoint from the cycles. We allow $u = v$ in which case the path has length zero.

We show that the expected number of such subgraphs is less than ϵ. The details appear in the full version of the paper. □

4 Proof of Theorem 1.1(b)

In this section we turn to the critical range of Q; i.e. $-\lambda n^{-1/3}R^{2/3} \leq Q \leq \lambda n^{-1/3}R^{2/3}$. We will bound the probability that the size of the largest component is too big. Without loss of generality, we can assume that $\lambda > \frac{1600}{\zeta}$. Our proof follows along the same lines as that of Theorem 1 (see also Theorem 7) of [34].

We wish to show that there exists a constant $B > 1$ such that with probability at least $1 - \epsilon$, the largest component has size at most $Bn^{2/3}R^{-1/3}$. To do so, we set $T := n^{2/3}R^{-1/3}$ and bound the probability that our branching process starting at a given vertex v does not return to zero within T steps. We do this by considering a stopping time that is at most the minimum of T and the first return to zero.

Lemma 2.2 yields that, with high probability, $|Q_t - Q| \leq \frac{1}{2}|Q| + \frac{800}{\zeta}n^{-1/3}R^{2/3}$ for every $t \leq T$. Since we assume $\lambda > \frac{1600}{\zeta}$, this implies $|Q_t| \leq 2\lambda n^{-1/3}R^{2/3}$. In order to assume that this bound always holds, we add it to our stopping time conditions, along with a similar condition for the concentration of R.

It will be convenient to assume that Y_t is bounded by $H := \frac{1}{12\lambda}n^{1/3}R^{1/3}$, so we add $Y_t \geq H$ to our stopping time conditions. Specifically, we define

$$\begin{aligned}\gamma := \min\{t \quad : \quad & (Y_t = 0), (Y_t \geq H),\\ & (|Q_t| > 2\lambda n^{-1/3}R^{2/3}),\\ & (|R_t - R| > R/2) \text{ or } (t = T)\}.\end{aligned}$$

Since $\Delta \leq n^{1/3}R^{1/3}/\ln n$, we have $T < \frac{\zeta}{400}\frac{n}{\Delta}$ for n sufficiently large. So Lemmas 2.1 and 2.2 imply that, with high probability, we will not have $|Q_\gamma| > 2\lambda n^{-1/3}R^{2/3}$ or $|R_\gamma - R| > R/2$. So by upper bounding $\mathbf{Pr}(Y_\gamma \geq H)$ and $\mathbf{Pr}(\gamma = T)$, we can obtain a good lower bound on $\mathbf{Pr}(Y_t = 0)$ which, in turn, is a lower bound on Y_t reaching zero before reaching H.

For $t \leq \gamma$, we have $|Q_{t-1}| \leq 2\lambda n^{-1/3}R^{2/3}$ and so:

(4.5) $$H|Q_{t-1}| \leq \tfrac{1}{6}R$$

For $t \leq \gamma$, we also have $Y_{t-1} > 0$ and so $\mathbb{E}(\eta_t)$ and $\mathbb{E}(\eta_t^2)$ are as in (2.1) and (2.2). We also have $R_{t-1} \geq \frac{1}{2}R$ and (4.5). When $|x|$ is sufficiently small we have $e^{-x} \geq 1 - x + x^2/3$. So for n sufficiently large, $|\eta_t/H| \leq (2+\Delta)/H < (\ln n)^{-1}$ is small enough to yield:

$$\begin{aligned}\mathbb{E}[e^{-\eta_t/H}|C_{t-1}] &\geq 1-\mathbb{E}[\frac{\eta_t}{H}|C_{t-1}]+\frac{1}{3}\mathbb{E}[\frac{\eta_t^2}{H^2}|C_{t-1}] \\ &= 1-\frac{Q_{t-1}}{H}+\frac{R_{t-1}}{3H^2} \\ &\geq 1-\frac{R}{6H^2}+\frac{R}{6H^2}=1.\end{aligned}$$

This shows that $e^{-Y_{\min(t,\gamma)}/H}$ is a submartingale, and so we can apply the Optional Stopping Theorem with stopping time $\tau := \gamma$. As $Y_{\gamma-1} \leq H$, we have $Y_\gamma \leq H+\Delta < 2H$. Recalling that we begin our branching process at vertex v and applying $x/4 \leq 1-e^{-x}$, for $0 \leq x \leq 2$, we have:

$$e^{-d_v/H}=e^{-Y_0/H} \leq \mathbb{E}e^{-Y_\gamma/H} \leq \mathbb{E}\left[1-\frac{Y_\gamma}{4H}\right],$$

which, using the fact that for $x > 0$, $e^{-x} \geq 1-x$, implies

$$\mathbb{E}[Y_\gamma] \leq 4H(1-e^{-d_v/H}) \leq 4d_v. \tag{4.6}$$

In particular

$$\Pr[Y_\gamma \geq H] \leq \frac{4d_v}{H}. \tag{4.7}$$

Now we turn our attention to $\mathbf{Pr}(\gamma = T)$. We begin by bounding:

$$\begin{aligned}\mathbb{E}[Y_t^2-Y_{t-1}^2|C_{t-1}] &= \mathbb{E}[(\eta_t+Y_{t-1})^2-Y_{t-1}^2|C_{t-1}] \\ &= \mathbb{E}[\eta_t^2|C_{t-1}]+ \\ &\quad 2\mathbb{E}[\eta_t Y_{t-1}|C_{t-1}].\end{aligned}$$

Now if $Y_{t-1} > 0$, then $\mathbb{E}[\eta_t|C_{t-1}] = Q_{t-1}$ and so $\mathbb{E}[\eta_t Y_{t-1}|C_{t-1}] = Q_{t-1}Y_{t-1}$, and if $Y_{t-1} = 0$, then $\mathbb{E}[\eta_t Y_{t-1}|C_{t-1}] = 0$. Also, for $t \leq \gamma$, we must have $Y_{t-1} < H$, $R_{t-1} \geq \frac{1}{2}R$ and (4.5) so:

$$\begin{aligned}\mathbb{E}[Y_t^2-Y_{t-1}^2|C_{t-1}] &\geq R_{t-1}+2H\min(Q_{t-1},0) \\ &\geq \frac{R}{2}-\frac{R}{3}=\frac{R}{6}.\end{aligned}$$

Thus $Y_{\min(t,\gamma)}^2 - \frac{1}{6}R\min(t,\gamma)$ is a submartingale, and so by the Optional Stopping Theorem:

$$\mathbb{E}\left[Y_\gamma^2-\frac{R\gamma}{6}\right] \geq Y_0^2 = d_v^2 \geq 0.$$

This, together with (4.6) and the fact (derived above) that $Y_\gamma \leq 2H$, implies that

$$\mathbb{E}\gamma \leq \frac{6}{R}\mathbb{E}Y_\gamma^2 \leq \frac{12H}{R}\mathbb{E}Y_\gamma \leq \frac{48Hd_v}{R},$$

showing

$$\Pr[\gamma = T] \leq \frac{48Hd_v}{RT}. \tag{4.8}$$

We conclude from (4.7), (4.8), and Lemmas 2.1 and 2.2 that, for n sufficiently large,

$$\begin{aligned}\Pr[|\mathcal{C}_v| \geq T] &\leq \Pr[Y_\gamma > H]+\Pr[\gamma = T] \\ &\quad +\Pr[|Q_t| > 2\lambda n^{-1/3}R^{2/3}] \\ &\quad +\Pr[|R_\gamma - R| > R/2] \\ &\leq \frac{4d_v}{H}+\frac{48Hd_v}{RT}+Tn^{-10}+Tn^{-10} \\ &\leq 48\lambda n^{-1/3}R^{-1/3}d_v \\ &\quad +\frac{48n^{1/3}R^{1/3}d_v}{12\lambda n^{2/3}R^{2/3}}+2Tn^{-10} \\ &< 50\lambda n^{-1/3}R^{-1/3}d_v.\end{aligned}$$

For some constant $B \geq 1$, let N be the number of vertices lying in components of size at least $K := Bn^{2/3}R^{-1/3} \geq T$. An easy argument yields $\sum_v d_v = 2|E| < 3n$ (see the full version of the paper). Therefore:

$$\begin{aligned}\Pr[|\mathcal{C}_{\max}| \geq K] &\leq \Pr[N \geq K] \leq \frac{\mathbb{E}[N]}{K} \\ &\leq \frac{1}{K}\sum_{v\in V}\Pr[\mathcal{C}_v \geq K] \\ &\leq \frac{1}{K}\sum_{v\in V}\Pr[\mathcal{C}_v \geq T] \\ &\leq \frac{1}{K}\sum_{v\in V}50\lambda n^{-1/3}R^{-1/3}d_v \\ &= \frac{50\lambda}{nB}\sum_v d_v < \frac{150\lambda}{B},\end{aligned}$$

which is less than ϵ for B sufficiently large. This proves that Theorem 1.1(b) holds for a random configuration. Proposition 2.1 implies that it holds for a random graph. □

5 Proof of Theorem 1.1(a)

In this section we bound the probability that the size of the largest component is too small when Q is in the critical range. Our proof follows along the same lines as that of Theorem 2 of [34].

Recall that we have $-\lambda n^{2/3}R^{2/3} \leq Q \leq \lambda n^{-1/3}R^{2/3}$. We can assume that $\lambda > \frac{1600}{\zeta}$. We wish to show that there exists a constant $A > 0$ such that with probability at least $1-\epsilon$, the largest component has size at least $An^{2/3}R^{-1/3}$.

We will first show that, with sufficiently high probability, our branching process reaches a certain value h. Then we will show that, with sufficiently high probability, it will take at least $An^{2/3}R^{-1/3}$ steps for it to get from h to zero, and thus there must be a component of that size.

We set $T_1 := n^{2/3}R^{-1/3}$ and $T_2 := An^{2/3}R^{-1/3}$. For $t \leq T_1 + T_2 \leq 2n^{2/3}R^{-1/3}$ (for $A \leq 1$), Lemma 2.2 yields that, with high probability, $|Q_t - Q| \leq \frac{1}{2}|Q| + \frac{800}{\zeta}n^{-1/3}R^{2/3}$ and thus (since $\lambda > \frac{1600}{\zeta}$)

$$Q_t \geq -2\lambda n^{-1/3}R^{2/3}.$$

We set

$$h := A^{1/4}n^{1/3}R^{1/3}$$

so that if $Q_t \geq -2\lambda n^{-1/3}R^{2/3}$ and $A < (16\lambda)^{-4}$ then

$$hQ_t \geq -2\lambda A^{1/4}R \geq -\frac{R}{8}. \tag{5.9}$$

We start by showing that Y_t reaches h, with sufficiently high probability. To do so, we define τ_1 analogously to γ from Section 4, except that we allow Y_t to return to zero before $t = \tau_1$.

$$\begin{aligned}\tau_1 = \min\{t \quad : \quad & (Y_t \geq h), (Q_t < -2\lambda n^{-1/3}R^{2/3}), \\ & (|R_t - R| > R/2), \text{ or } (t = T_1)\}.\end{aligned}$$

We wish to show that, with sufficiently high probability, we get $Y_{\tau_1} \geq h$. We know that the probability of $Q_{\tau_1} < -2\lambda n^{-1/3}R^{2/3}$ or $|R_{\tau_1} - R| > R/2$ is small by Lemmas 2.1 and 2.2. So it remains to bound $\mathbf{Pr}(\tau_1 = T_1)$. For $t \leq \tau_1$, if $Y_{t-1} > 0$, then by (2.1), (2.2), (5.9) and the fact that $Y_{t-1} < h$:

$$\begin{aligned}\mathbb{E}[Y_t^2 - Y_{t-1}^2|C_{t-1}] &= \mathbb{E}[\eta_t^2|C_{t-1}] + \\ &\quad 2\mathbb{E}[\eta_t Y_{t-1}|C_{t-1}] \\ &\geq R_{t-1} + 2h\min(Q_{t-1}, 0) \\ &\geq \frac{R}{2} - \frac{R}{4} \geq \frac{R}{4},\end{aligned}$$

Also if $Y_{t-1} = 0$, then by (2.4) we have

$$\mathbb{E}[Y_t^2 - Y_{t-1}^2|C_{t-1}] = \mathbb{E}[\eta_t^2|C_{t-1}] \geq \frac{R_{t-1}}{2} \geq \frac{R}{4}.$$

Thus $Y^2_{\min(t,\tau_1)} - \frac{1}{4}R\min(t,\tau_1)$ is a submartingale, so we can apply the Optional Stopping Theorem to obtain:

$$\mathbb{E}Y_{\tau_1}^2 - \frac{R}{4}\mathbb{E}\tau_1 \geq Y_0^2 \geq 0,$$

and as $Y_{\tau_1} \leq 2h$,

$$\mathbb{E}\tau_1 \leq \frac{4}{R}\mathbb{E}Y_{\tau_1}^2 \leq \frac{16h^2}{R}.$$

Hence

$$\Pr[\tau_1 = T_1] \leq \frac{16h^2}{RT_1}. \tag{5.10}$$

By the bound $\Delta \leq n^{1/3}R^{1/3}/\ln n$, we have $T_1 + T_2 < \frac{\zeta}{400}\frac{n}{\Delta}$. So Lemmas 2.1 and 2.2 imply that for sufficiently large n,

$$\begin{aligned}\Pr[Y_{\tau_1} < h] &\leq \Pr[\tau_1 = T_1] \\ &\quad + \Pr[Q_{\tau_1} < -2\lambda n^{-1/3}R^{2/3}] \\ &\quad + \Pr[|R_{\tau_1} - R| > R/2] \\ &\leq \frac{16h^2}{RT_1} + 2T_1 n^{-10} < 20\sqrt{A}. \qquad (5.11)\end{aligned}$$

This shows that with probability at least $1 - 20\sqrt{A}$, Y_t will reach h within T_1 steps. If it does reach h, then the largest component must have size at least h, which is not as big as we require. We will next show that, with sufficiently high probability, it takes at least T_2 steps for Y_t to return to zero, hence establishing that the component being exposed has size at least T_2, which is big enough to prove the theorem.

Let Θ_h denote the event that $Y_{\tau_1} \geq h$. Note that whether Θ_h holds is determined by C_{τ_1}. Much of what we say below only holds if C_{τ_1} is such that Θ_h holds.

Define

$$\begin{aligned}\tau_2 = \min\{s \quad : \quad & (Y_{\tau_1+s} = 0), \\ & (Q_{\tau_1+s} < -2\lambda n^{-1/3}R^{2/3}), \\ & (|R_{\tau_1+s} - R| > R/2), \text{ or } (s = T_2)\}.\end{aligned}$$

We wish to show that, with sufficiently high probability, we get $\tau_2 = T_2$ as this implies $Y_{\tau_1+T_2-1} > 0$. We know that the probability of $Q_{\tau_1+\tau_2} < -2\lambda n^{-1/3}R^{2/3}$ or $|R_{\tau_1+\tau_2} - R| > R/2$ is small by Lemmas 2.1 and 2.2. So it remains to bound $\mathbf{Pr}[Y_{\tau_1+s} = 0]$.

Suppose that Θ_h holds. It will be convenient to view the random walk back to $Y_t = 0$ as a walk from 0 to h rather than from h to 0; and it will also be convenient if that walk never drops below 0. So we define $M_s = h - \min\{h, Y_{\tau_1+s}\}$, and thus $M_s \geq 0$ and $M_s = h$ iff $Y_{\tau_1+s} = 0$. If $0 < M_{s-1} < h$, then $M_{s-1} = h - Y_{\tau_1+s-1}$ and since $M_s \leq |h - Y_{\tau_1+s}|$, we have:

$$\begin{aligned}M_s^2 - M_{s-1}^2 &\leq (h - Y_{\tau_1+s})^2 - (h - Y_{\tau_1+s-1})^2 \\ &= 2h(Y_{\tau_1+s-1} - Y_{\tau_1+s}) \\ &\quad + Y_{\tau_1+s}^2 - Y_{\tau_1+s-1}^2 \\ &= \eta_{\tau_1+s}(Y_{\tau_1+s} + Y_{\tau_1+s-1} - 2h) \\ &= \eta_{\tau_1+s}(\eta_{\tau_1+s} - 2M_{s-1}) \\ &= \eta_{\tau_1+s}^2 - 2\eta_{\tau_1+s}M_{s-1}. \qquad (5.12)\end{aligned}$$

If $M_{s-1} = 0$, then $Y_{\tau_1+s-1} \geq h$ and so

$$M_s^2 - M_{s-1}^2 = M_s^2 \leq \eta_{\tau_1+s}^2. \tag{5.13}$$

Consider any C_{τ_1+s-1} for which Θ_h holds. For $1 \leq s \leq \tau_2$, we have $M_{s-1} < h$ and by (2.4) we have $\mathbb{E}[\eta_{\tau_1+s}^2|C_{\tau_1+s-1}] = R_{\tau_1+s} \leq \frac{3}{2}R$ since $|R_{\tau_1+s} - R| \leq R/2$. Applying those, along with (5.12), (5.13) and (5.9) we obtain that

$$\mathbb{E}[M_s^2 - M_{s-1}^2|C_{\tau_1+s-1}]$$

is bounded from above by

$$\begin{aligned}
&\max(\mathbb{E}[\eta_{\tau_1+s}^2|C_{\tau_1+s-1}],\\
&\mathbb{E}[\eta_{\tau_1+s}^2 - 2\eta_{\tau_1+s}M_{s-1}|C_{\tau_1+s-1}])\\
&\leq \max\left(\frac{3R}{2}, \frac{3R}{2} - 2hQ_{\tau_1+s-1}\right)\\
&\leq \frac{3R}{2} + \frac{R}{4} < 2R.
\end{aligned}$$

So for any C_{τ_1} for which Θ_h holds, $M_{\min(s,\tau_2)}^2 - 2R\min(s,\tau_2)$ is a supermartingale, and the Optional Stopping Theorem yields:

$$\mathbb{E}[M_{\tau_2}^2 - 2R\tau_2] \leq \mathbb{E}M_0^2 = 0.$$

This, along with the fact that $\tau_2 \leq T_2$ yields:

$$\mathbb{E}M_{\tau_2}^2 \leq 2R\mathbb{E}\tau_2 \leq 2T_2R.$$

By (5.11) we have that for any event E, $\mathbf{Pr}(E|\Theta_h) \leq \mathbf{Pr}(E)/\mathbf{Pr}(\Theta_h) \leq \mathbf{Pr}(E)/(1-20\sqrt{A})$. Hence Lemmas 2.1 and 2.2 yield that for n sufficiently large $\Pr[\tau_2 < T_2|\Theta_h]$ is at most:

$$\begin{aligned}
&\Pr[M_{\tau_2} \geq h|\Theta_h] +\\
&\Pr[Q_{\tau_1+\tau_2} < -2\lambda n^{-1/3}R^{2/3}|\Theta_h] +\\
&\Pr[|R_{\tau_1+\tau_2} - R| > R/2|\Theta_h]\\
&\leq \frac{\mathbb{E}M_{\tau_2}^2}{h^2} + \frac{2T_2n^{-10}}{1-20\sqrt{A}}\\
&\leq \frac{2T_2R}{h^2} + \frac{2T_2n^{-10}}{1-20\sqrt{A}}\\
&\leq \frac{3T_2R}{h^2}.
\end{aligned}$$

Combining this with (5.11) we conclude

$$\begin{aligned}
\Pr[|\mathcal{C}_{\max}| < T_2] &\leq \Pr[\tau_2 < T_2]\\
&\leq \Pr[Y_{\tau_1} < h] + \Pr[\tau_2 < T_2|\Theta_h]\\
&\leq 20\sqrt{A} + \frac{3T_2R}{h^2} = 23\sqrt{A} < \epsilon,
\end{aligned}$$

for $A < (\frac{\epsilon}{23})^2$. (Recall that we also require $A < (16\lambda)^{-4}$.) This proves that Theorem 1.1(a) holds for a random configuration. Proposition 2.1 implies that it holds for a random graph. □

Proof of Theorem 1.3 We can apply essentially the same argument as for Theorem 1.1(a). In fact, the argument is a bit simpler here as we will always have the drift $Q_t > 0$. The details are in the full version of the paper. □

References

[1] W. Aiello, F. Chung and L. Lu. *A random graph model for massive graphs.* Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing (2000), 171–180.

[2] R. Arratia, A. Barbour and S. Tavaré. *Logarithmic Combinatorial Structures: a Probabilistic Approach.* European Math Society, Zurich (2003).

[3] W. Barthel, A. Hartmann and M. Weigt. *Phase transition and finite-size scaling in the vertex-cover problem.* Comp. Phys. Comm. **169** (2005), 234 - 237.

[4] E.Bender and R. Canfield. *The asymptotic number of labelled graphs with given degree sequences.* J. Comb. Th. (A) **24** (1978), 296 - 307.

[5] S. Bhamidi, R. van der Hofstad and J. van Leeuwaarden. *Novel scaling limits for critical inhomogeneous random graphs.* Preprint.

[6] S. Bhamidi, R. van der Hofstad and J. van Leeuwaarden. *Scaling limits for critical inhomogeneous random graphs with finite third moments.* Preprint.

[7] B. Bollobás. *A probabilistic proof of an asymptotic formula for the number of labelled graphs.* Europ. J. Comb. **1** (1980), 311 - 316.

[8] B. Bollobás. *The evolution of random graphs.* Trans. Am. Math. Soc. **286** (1984), 257 - 274.

[9] B. Bollobás. *Random Graphs.* 2nd Edition. Cambridge University Press (2001).

[10] B. Bollobás, C. Borgs, J. Chayes, J. Kim and D. Wilson. *The scaling window of the 2-SAT transition.* Rand. Struc. & Alg. **18** (2001), 201 - 256.

[11] B. Bollobás, S. Jansen and O. Riordan. *The phase transition in inhomogeneous random graphs.* Rand. Struc & Alg. **31** (2007), 3 – 122.

[12] C. Borgs, j. Chayes and B. Pittel. *Sharp threshold and scaling window for the integer partitioning problem.* Proceedings of the 33rd annual ACM Symposium on the Theory of Computing (STOC), 330 - 336 (2001).

[13] F. Chung and L. Lu. *Concentration inequalities and martingale inequalities: a survey.* Internet Math. **3** (2006), 79 - 127.

[14] V. Chvátal and B. Reed. *Mick gets some (the odds are on his side).* Proceedings 33rd Annual Symposium on Foundations of Computer Science, 620 - 627 (1992).

[15] C. Cooper and A. Frieze. *The size of the largest strongly connected component of a random digraph with a given degree sequence.* Combin. Prob. & Comp **13** (2004), 319 - 338.

[16] W. Fernandez de la Vega. *On random 2-SAT.* manuscript (1992).

[17] P. Erdős and A. Rényi. *On the evolution of random graphs.* Magayar Tud. Akad. Mat. Kutato Int. Kozl. **5** (1960), 17 - 61.

[18] N. Fountoulakis and B. Reed. *Critical conditions for the emergence of a giant component.* Proceedings of Eurocomb 2009. (Journal version in preparation.)

[19] A. Goerdt. *A threshold for unsatisfiability.* J. Comp. and System Sci. **53** (1996), 469 - 486.

[20] A. Hartmann and M. Weigt. *Phase Transitions in Combinatorial Optimization Problems.* Wiley, Berlin (2005).

[21] R. van der Hofstad. *Critical behaviour in inhomogeneous random graphs.* Preprint.

[22] S. Jansen. *The probability that a random multigraph is simple.* Combin. Prob. & Comp. **18** (2009), 205 - 225.

[23] S. Jansen and M. Luczak. *A new approach to the giant component problem.* Rand. Struc & Alg. (to appear).

[24] M. Kang and T. Seierstad. *The critical phase for random graphs with a given degree sequence.* Combin. Prob. & Comp **17** (2008), 67 - 86.

[25] R. Karp. *The transitive closure of a random digraph.* Rand. Struc & Alg. **1** (1990), 73 - 94.

[26] M. Leone, F. Ricci-Tersenghi and R. Zecchina. *Phase coexistence and finite size scaling in random combinatorial problems.* J. Phys. A **34** (2001) 4615 - 4626.

[27] D. Levin, Y. Peres and E. Wilmer. *Markov Chains and Mixing Times.* American Math. Soc. (2008).

[28] T. Łuczak. *Component behaviour near the critical point of the random graph process.* Rand. Struc. & Alg. **1** (1990), 287 - 310.

[29] T. Łuczak, B. Pittel and J. Weirman. *The structure of a random graph at the point of the phase transition.* Trans. Am. Math. Soc. **341** (1994), 721 - 748.

[30] B. McKay and N. Wormald. *Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$.* Combinatorica **11** (1991), 369 - 382.

[31] S. Mertens, M. Mezard and R. Zecchina. *Threshold values of random k-SAT from the cavity method.* Rand. Struc. & Alg. **28** (2006), 340 - 373.

[32] M. Molloy and B. Reed. *A critical point for random graphs with a given degree sequence.* Rand. Struc. & Alg. **6** (1995), 161 - 180.

[33] M. Molloy and B. Reed. *The size of the largest component of a random graph on a fixed degree sequence.* Combin. Prob. & Comp **7** (1998), 295 - 306.

[34] A. Nachmias and Y. Peres. *The critical random graph with martingales.* Israel J. Math. (to appear).

[35] A. Nachmias and Y. Peres. *Component sizes of the random graph outside the scaling window.* Latin Am. J. Prob. and Math. Stat. **3** (2007), 133 - 142.

[36] M. Newman, A. Barabási and D. Watts. *The Structure and Dynamics of Networks.* Princeton University Press (2006).