A Bound on the Strong Chromatic Index of a Graph

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November 2, 2000

Abstract

We show that the strong chromatic index of a graph with maximum degree Δ is at most $(2-\epsilon)\Delta^2$, for some $\epsilon > 0$. This answers a question of Erdős and Nešetřil.

1 Introduction

A strong edge-colouring of a (simple) graph, G, is a proper edge-colouring of G with the added restriction that no edge is adjacent to two edges of the same colour. (Note that in any proper edge-colouring of G, no edge is adjacent to three edges of the same colour.) Equivalently, it is a proper vertex-colouring of $L(G)^2$, the square of the line graph of G[1]. The strong chromatic index of G, $s\chi'(G)$ is the least integer k such that G has a strong edge-colouring using k colours.

If G has maximum degree Δ, then trivially $s\chi'(G) \leq 2\Delta^2 - 2\Delta + 1$, as $L(G)^2$ has maximum degree at most $2\Delta^2 - 2\Delta$. In 1985, Erdős and Nešetřil (see [5]) asked if there is any $\epsilon > 0$ such that for every such G, $s\chi'(G) \leq (2-\epsilon)\Delta^2$. They pointed out that by multiplying the vertices of the 5-cycle, one can obtain a graph, G, with arbitrarily large Δ for which $s\chi'(G) = \frac{5}{4}\Delta^2$, and conjectured that, in fact, for every G with maximum degree Δ, $s\chi'(G) \leq \frac{5}{4}\Delta^2$. For other work on this and related problems, see [2], [3], [6], [7] and [8].

In this paper we answer their question in the affirmative, with $\epsilon = .002$:

Theorem 1 *If G has maximum degree Δ sufficiently large, then $s\chi'(G) \leq 1.998\Delta^2$.*

Our proof is probabilistic and we make no attempt to find the best possible value of ϵ. In fact, as described in Section 4, a simple modification of our arguments will yield an improvement of $\epsilon > .01$. However, it seems that our techniques are not sufficient to find ϵ near $\frac{3}{4}$.

To prove Theorem 1, we show the following. If H is a graph with maximum degree at most X, and if for some constant $\delta > 0$, no vertex of H has more than $(1-\delta)\binom{X}{2}$ edges in the subgraph induced by its neighbourhood, then the chromatic number $\chi(H) \leq (1-\gamma)X$, where $\gamma = \gamma(\delta) > 0$.

The idea behind the proof of this is simple. We will colour H with $(1-\gamma)X$ colours as follows. First we assign each vertex a uniformly random colour. Next, we uncolour every vertex which has a neighbour of the same colour. We show that with positive probability, each vertex has at least γX repeated colours in its neighbourhood, and so we can complete the colouring of H in a simple greedy manner.

Theorem 1 then follows by showing that $L(G)^2$ satisfies these prerequisites with $X = 2\Delta^2$ and $\delta = \frac{1}{36}$, and that $\gamma(\frac{1}{36}) \geq .001$.

[1] The line graph of G, $L(G)$ has vertex set $E(G)$, and two vertices are adjacent in $L(G)$ iff the corresponding edges are incident in G. The square of a graph H, H^2, has vertex set $V(H)$, and two vertices are adjacent iff they are of distance at most two in H.

2 Preliminaries

We will need two tools of the probabilistic method. The first is due to Lovász.

The Local Lemma [4] *Suppose* $\mathcal{A} = A_1, \ldots, A_n$ *is a list of random events such that for each* i, $\mathbf{Pr}(A_i) \leq p$ *and* A_i *is mutually independent of all but at most* d *other events in* $\mathcal{A}$. *If* $\mathrm{e}p(d+1) < 1$ *then* $\mathbf{Pr}(\wedge_{i=1}^{n} \bar{A}_i) > 0$.

For the second tool, we introduce the following definition. Suppose $\Omega = \prod_{i=1}^{\omega} \Omega_i$ is a product probability space, and denote by x_i the ith coordinate of an $x \in \Omega$. For any $A \subset \Omega$, we define A_t by saying $y \in A_t$ iff for any $\alpha_1, \ldots, \alpha_\omega \in \mathbf{R}$, there exists $x \in A$ such that

$$\sum_{x_i \neq y_i} \alpha_i < t \sqrt{\sum_{i=1}^{\omega} \alpha_i^2}.$$

Talagrand's Inequality [11] *For any* $A \subset \Omega$, $t > 0$,

$$\mathbf{Pr}(A) \times (1 - \mathbf{Pr}(A_t)) < \mathrm{e}^{-t^2/4}.$$

In other words, either $\mathbf{Pr}(A)$ is small or $\mathbf{Pr}(\overline{A_t})$ is small. This yields the following corollary. We say that $h : \Omega \to \mathbf{R}$ is *Lipschitz* if $|h(x) - h(y)| \leq 1$ for all $x, y \in \Omega$ which differ in only one coordinate. For any $f : \mathbf{R} \to \mathbf{R}$, we say that h is f*-certifiable* if whenever $h(x) \geq s$, there is a set of at most $f(s)$ coordinates of x which certify that $h(x) \geq s$; more formally there exists $I \subseteq \{1, \ldots, \omega\}$, $|I| \leq f(s)$ such that if for any $y \in \Omega$, $y_i = x_i$ for each $i \in I$ then $h(y) \geq s$.

Corollary 1 *If* $h : \Omega \to \mathbf{R}$ *is Lipschitz and* f*-certifiable, then for any* b *and any* $t \geq 0$

$$\mathbf{Pr}(h(x) < b - t\sqrt{f(b)}) \times \mathbf{Pr}(h(x) > b) < \mathrm{e}^{-t^2/4}.$$

In other words, with high probability h lies in a relatively small range. For a proof and further discussion of this corollary, see [10].

For any $v \in V(G)$ we denote by $N_G(v)$ the neighbourhood of v in G, sometimes writing $N(v)$ if no confusion arises. For any subsets $A, B \subseteq V(G)$, we denote by $E(A, B)$ the set of edges with one endpoint in each of A, B, and we set $E(A) = E(A, A)$. We also denote by $\deg_A(v)$ the number of neighbours v has in A. Sometimes, in order to avoid confusion, we refer to an edge of a graph G as a G*-edge*. All graphs are assumed to be simple. We only claim all statements to hold for Δ or X sufficiently large.

3 Details

We will prove the following two Lemmas.

Lemma 1 *If G has maximum degree Δ, then for each $e \in V(L(G)^2)$, $N_{L(G)^2}(e)$ has at most $(1-\frac{1}{36})\binom{2\Delta^2}{2}$ $L(G)^2$-edges.*

Lemma 2 *Consider any $\delta, \gamma > 0$ such that $\gamma < \frac{\delta}{2(1-\gamma)}\mathrm{e}^{-3/(1-\gamma)}$. Suppose that H is a graph with maximum degree at most X (sufficiently large), such that for each $v \in V(H)$, $N(v)$ has at most $(1-\delta)\binom{X}{2}$ edges. Then $\chi(H) \leq (1-\gamma)X$.*

Note that Theorem 1 follows immediately from Lemmas 1 and 2, as $\gamma = .001$ satisfies the conditions of Lemma 2 with $\delta = \frac{1}{36}$ and $X = 2\Delta^2$.

Proof of Lemma 1: Without loss of generality, we will assume that G is Δ-regular throughout this proof. Set $H = L(G)^2$.

Consider any G-edge $e = (u_1, u_2)$. Set $A = N_G(u_1), B = N_G(u_2)$, and $C = N_G(A) \cup N_G(B) - (A \cup B)$.

Case 1. $|E_G(A \cup B)| + \Delta|A \cap B| > \Delta^2/30$.

In this case, $|N_H(e)| < (2 - \frac{1}{30})\Delta^2$, and so $N_H(e)$ has at most $(2 - \frac{1}{15} + \frac{1}{1800})\Delta^4 < (1 - \frac{1}{36})\binom{2\Delta^2}{2}$ H-edges.

Case 2. $\sum_{c \in C} \deg_{A\cup B}(c) \times (\Delta - \deg_{A\cup B}(c)) > (2 - \frac{1}{9})\Delta^4$.

For any $e_1 \in N_H(e)$, the number of G-edges in $N_H(e)$ to which e_1 is adjacent in H is at most $2\Delta^2$ minus the number of G-paths of length at most 3, with first edge e_1 and last edge not in $N_H(e)$. Every G-path (a, c, x), where $a \in A \cup B$, $c \in C$, $x \notin A \cup B$, contributes Δ of the aforementioned paths and so the total number of these paths is at least $\frac{1}{9}\Delta^4$ and so the total number of H-edges in $N_H(e)$ is at most $(1 - \frac{1}{36})\binom{2\Delta^2}{2}$.

Case 3. $|E_G(A \cup B)| + \Delta|A \cap B| \leq \Delta^2/30$ and $\sum_{c \in C} \deg_{A\cup B}(c) \times (\Delta - \deg_{A\cup B}(c)) \leq (2 - \frac{1}{9})\Delta^4$.

In this case, we will bound the number of H-edges in $N_H(e)$ by counting the number of 4-cycles in G using G-edges from $E_G(A \cup B, C)$. Note that the number of H-edges in $N_H(e)$ is at most $2\Delta^4$ minus twice the number of these cycles, as the degree in H of any G-edge in $E(A \cup B, C)$ is at most $2\Delta^2$ minus the number of these cycles in which it lies.

For each $c_1, c_2 \in C$, denote by $w(c_1, c_2)$ the number of common neighbours c_1, c_2 have in $A \cup B$. Note that the number of these 4-cycles is at least

$$\sum_{c_1, c_2 \in C} \binom{w(c_1, c_2)}{2}.$$

Let $C^{'} = \{c \in C | \deg_{A \cup B}(c) \geq \frac{2}{3}\Delta\}$. Note that $|E_G(A \cup B, C^{'})| \geq (\frac{5}{3} - \frac{1}{15})\Delta^2 = \frac{8}{5}\Delta^2$, as otherwise we are in one of the previous cases.

Therefore, by the Cauchy-Schwartz Inequality,

$$\begin{aligned}
\sum_{c_1, c_2 \in C^{'}} w(c_1, c_2) &= \sum_{a \in A \cup B} \binom{\deg_{C^{'}}(a)}{2} \\
&\geq 2\Delta \times \binom{\frac{4}{5}\Delta}{2} \\
&= \frac{16}{25}\Delta^3 + \mathrm{O}(\Delta^2).
\end{aligned}$$

Also, $|C^{'}| \leq 3\Delta$, and so

$$\begin{aligned}
\sum_{c_1, c_2 \in C^{'}} \binom{w(c_1, c_2)}{2} &\geq \binom{3\Delta}{2} \times \binom{\frac{16}{25}\Delta^3 / \binom{3\Delta}{2}}{2} + \mathrm{O}(\Delta^3) \\
&= \frac{(16)^2}{9(25)^2}\Delta^4 + \mathrm{O}(\Delta^3) \\
&> \frac{1}{36}\Delta^4,
\end{aligned}$$

and therefore the number of edges in $N_H(e)$ is at most $(1 - \frac{1}{36})\binom{2\Delta^2}{2}$. $\square$

Proof of Lemma 2:

First note that we can assume H to be X-regular, as it is straightforward to construct an X-regular graph $H^{'}$, such that $H \subseteq H^{'}$ and $H^{'}$ satisfies the prerequisites of Lemma 2. For example, by taking many disjoint copies of H, and adding edges between copies of vertices of degree less than X in an appropriate manner.

Set $\zeta = \frac{\delta}{2(1-\gamma)}\mathrm{e}^{-3/(1-\gamma)}$. We will colour the vertices of H as described in the introduction. First, we assign to each vertex a uniformly random colour from amongst a set of $c = \lceil (1-\gamma)X \rceil$ colours. Next, we uncolour every vertex which is adjacent to another vertex of the same colour. For each vertex v, let A_v be the event that the difference between the number of vertices coloured in $N(v)$ and the number of colours used in $N(v)$ is less than $\zeta X - 14\sqrt{X \log X}$.

We will use the Local Lemma to show that with positive probability, A_v does not hold for any v. Note that if this is the case, then for sufficiently large X, we can complete our colouring of H in a simple greedy manner, since $\gamma < \zeta$.

Consider any vertex v. After the initial colouring (but before any vertices are uncoloured), we say that $u_1, u_2 \in N(v)$ are *compatible* if (i) they are nonadjacent, (ii) they have the same colour, (iii) neither has a neighbour with that colour, and (iv) no other vertex in $N(v)$ has that colour. Let C be the random variable denoting the number of compatible pairs of vertices in $N(v)$, and let D be the random variable denoting the number of compatible pairs which both retain their colours after the uncolouring stage.

To bound $\mathbf{Pr}(A_v)$, we will consider $A_v^{'}$ - the event that $D < \zeta X - 14\sqrt{X \log X}$. Note that if $A_v^{'}$ does not hold, then neither does A_v, and so $\mathbf{Pr}(A_v) \le \mathbf{Pr}(A_v^{'})$.

Note that any nonadjacent pair is compatible with probability at least $\frac{1}{c}(1-\frac{1}{c})^{3X-2} > \frac{2\zeta}{\delta X}$ and so $\mathbf{Exp}(D^{'}) \ge \zeta X$. We will use Corollary 1 to show that $D^{'}$ is highly concentrated around its expected value.

For $i \in V(G)$, we let x_i denote the colour of i, and so Ω_i has our colour set as domain, and the uniform distribution. Setting $h(x) = -\frac{1}{2}D^{'}$, we see that h is Lipschitz as changing the colour of any one vertex can affect the value of $D^{'}$ by at most 2. Furthermore, h is f-certifiable with $f(s) = 1.5X$ as the colouring of $N(v)$ along with the colours of at most $\frac{X}{2}$ vertices in $N(N(v))$ provide a certificate. Thus, we can apply Corollary 1 with $t = 7\sqrt{\log X}$ to show that for all $b < X$,

$$\mathbf{Pr}\left(h < b - 7\sqrt{X \log X}\right) \times \mathbf{Pr}(h > b) < \mathrm{e}^{-t^2/4} < X^{-10}.$$

Therefore there is some range θ of size at most $7\sqrt{X \log X}$, such that $\mathbf{Pr}(h \notin \theta) < X^{-5}$. Since $|h|$ is bounded above by $X/4$, it is straightforward to show that $\mathbf{Exp}(h) \in \theta$. And so $\mathbf{Pr}(A_v^{'}) \le \mathbf{Pr}(|D^{'} - \mathbf{Exp}(D^{'})| > 14\sqrt{X \log X}) < X^{-5}$.

Therefore $\mathbf{Pr}(A_v) \le X^{-5}$. Furthermore, each A_v is independent of all but at most X^4 other such events. Therefore, since $\mathrm{e} \times X^{-5} \times (X^4 + 1) \le 1$, by the Local Lemma $\mathbf{Pr}(\wedge_v \bar{A}_v) > 0$, and so there is a partial colouring which can be extended greedily into a $(1-\gamma)X$ colouring. $\square$

4 Remarks

It is worth noting that a simple modification of our methods, leads to an improvement of the constant in Theorem 1 to 1.99. Instead of uncolouring every vertex with a neighbour with the same colour, we assign a uniformly random real weight $w_v \in [0,1]$ to each vertex v, and uncolour a vertex if it has a neighbour with the same colour *and* a higher weight. We also allow D' to count compatible pairs with a few more vertices of the same colour in $N(v)$. It seems that the best constant that these methods can lead to is not much smaller than 1.9 which is far from the objective of 1.25.

The authors would like to thank two anonymous referees for their helpful comments.

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