CSC2556

Lecture 8

Leximin
Rent Division

Leximin (DRF)

Computational Resources

- Resources: Homogeneous divisible resources like CPU, RAM, or network bandwidth
- Valuations: Each player wants the resources in a fixed proportion (Leontief preferences)
- Example:
  - Player 1 requires (2 CPU, 1 RAM) for each copy of task
  - Indifferent between (4,2) and (5,2), but prefers (5,2.5)
  - “fractional” copies are allowed

Model

- Set of players $N = \{1, \ldots, n\}$
- Set of resources R, $|R| = m$

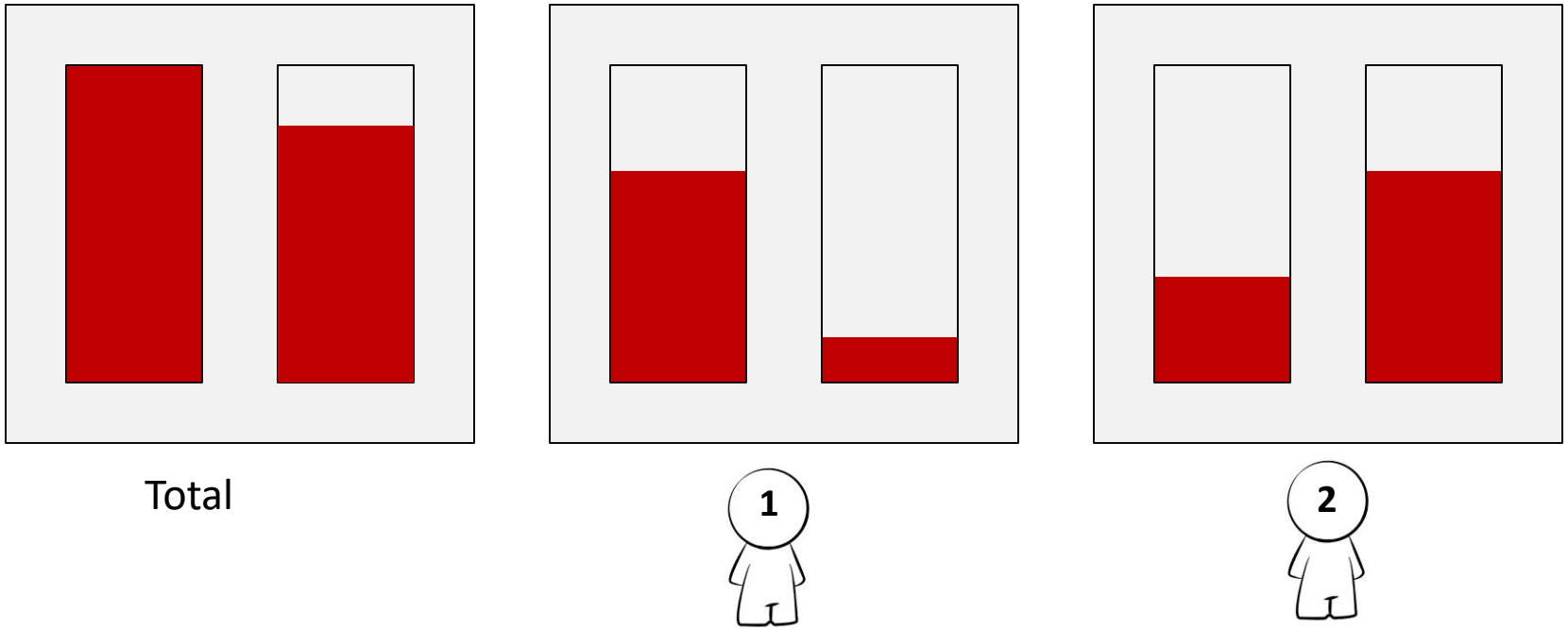
- Demand of player i is $d_i = (d_{i1}, \ldots, d_{im})$
  - $0 < d_{ir} \leq 1$ for every r, $d_{ir} = 1$ for some r
    - "For every 1% of the total available CPU you give me, I need 0.5% of the total available RAM"

- Allocation: $A_i = (A_{i1}, \ldots, A_{im})$ where A_{ir} is the fraction of available resource r allocated to i
  - Utility to player i : $u_i(A_i) = \min_{r \in R} A_{ir}/d_{ir}$.
  - We'll assume a non-wasteful allocation
    - Allocates resources proportionally to the demand.

Dominant Resource Fairness

- Dominant resource of i is r such that $d_{ir} = 1$
- Dominant share of i is A_{ir}, where r = dominant resource of i
- Dominant Resource Fairness (DRF) Mechanism
  - Allocate maximal resources while maintaining equal dominant shares.

DRF animated



Properties of DRF

- Envy-free: $u_i(A_i) \geq u_i(A_j), \forall i, j$
  - Why? [Note: EF no longer implies proportionality.]
- Proportionality: $u_i(A_i) \geq 1/n, \forall i$
  - Why?
- Pareto optimality (Why?)
- Group strategyproofness:
  - If a group of players manipulate, it can't be that none of them lose, and at least one of them gains.
  - We'll skip this proof.

The Leximin Mechanism

- Generalizes the DRF Mechanism
- Mechanism:
  - Choose an allocation A that
    - Maximizes $\min_i u_i(A_i)$
    - Among all minimizers, breaks ties in favor of higher second minimum utility.
    - Among all minimizers, breaks ties in favor of higher third minimum utility.
    - And so on…
- Maximizes the egalitarian welfare

The Leximin Mechanism

- DRF is the leximin mechanism
  - In the previous illustration, we didn't need tie-breaking because we assumed $d_{ir} > 0$ for every $i \in N, r \in R$.
  - In practice, not all the players need all the resources.
  - When $d_{ir} = 0$ is allowed, we need to continue allocating even after some agents are saturated.
    - Not all agents have equal dominant shares in the end.

- Theorem [Parkes, Procaccia, S '12]:
  - When $d_{ir} = 0$ is allowed, the leximin mechanism still retains all four properties (proportionality, envy-freeness, Pareto optimality, group strategyproofness).

A Note on Dynamic Settings

- We assumed that all agents are present from the start, and we want a one-shot allocation.
- Real-life environments are dynamic. Agents arrive and depart, and their demands change over time.

- Theorem [Kash, Procaccia, S '14]:
  - A dynamic version of the leximin mechanism satisfies proportionality, Pareto optimality, and strategyproofness along with a relaxed version of envy-freeness when agents arrive one-by-one.

A Note on Dynamic Settings

- Dynamic mechanism design
  - Designing fair, efficient, and game-theoretic mechanisms in dynamic environments is a relatively new research area, and we do not know much.
  - E.g., what if agents can depart, demands can change over time, or agents can submit and withdraw multiple jobs over time?
  - Lots of open questions!

Leximin (Dichotomous Matching)

Matching + Dichotomous Prefs

- Recall the stable matching setting of matching n men to n women.
  - We assumed ranked preferences, and showed that the Gale-Shapley algorithm produces a stable matching.
  - What if agent preferences weren't ranked?
- Suppose the men and women have dichotomous preferences over each other.
  - Each man finds a subset of women "acceptable" (utility 1), and the rest "unacceptable" (utility 0).
  - Same for women's preferences over men.

Matching + Dichotomous Prefs

- Dichotomous preferences induce a bipartite graph betwee men and women.
  - If a perfect matching exists, it's awesome.
  - What if there is no perfect matching?
    - Any deterministic matching unfairly gives 0 utility to some agents.
    - Solution: randomize!

- Under a random matching, utility to an agent = probability of being matched to an acceptable partner.

Matching + Dichotomous Prefs

- (Integral) Matching:
  - "Select" or "not select" each edge such that the number of selected edges incident on each vertex is at most 1.
- Fractional Matchings:
  - "Put a weight" on each edge such that the total weight of edges incident on each vertex is at most 1.
- Birkoff von-Neumann Theorem:
  - Every fractional matching can be "implemented" as a probability distribution over integral matchings.

Matching + Dichotomous Prefs

- Randomized leximin mechanism:
  - Compute the leximin fractional matching, and implement it as a distribution over integral matchings.
  - Both steps are doable in polynomial time!
- Theorem [Bogomolnaia, Moulin '04]:
  - The randomized leximin mechanism satisfies proportionality, envy-freeness, Pareto optimality, and group-strategyproofness (for both sides).
- In contrast: For ranked preferences, no algorithm can be strategyproof for both sides.

Matching with Capacities

- Proposition 39 in California
  - "Unused resources in public schools should be *fairly* allocated to local charter schools that desire them."
- Each charter school (agent) i wants d_i unused classrooms at one of the acceptable public schools (facilities) F_i.
  - If the demand is met, the charter school can relocate to the public school facility.
- Each facility j has c_j unused classrooms.
  - We assume facilities don't have preferences over agents.

Leximin (Classroom Allocation)

Model

Agents have demands

6

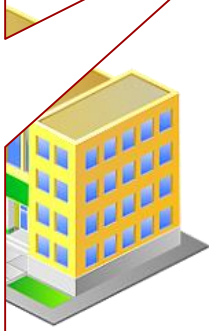
3

8

4

Preferences are dichotomous

2015/2016 request form: "provide a description of the district school site and/or general geographic area in which the charter school wishes to locate"



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Facilities have capacities

Number of unused classrooms

Leximin Strikes Again

- Utility of agent i under a randomized allocation = probability of being allocated d_i classrooms at one of the facilities in F_i.

- Theorem [Kurokawa, Procaccia, S '15]:
  - The randomized leximin mechanism satisfies proportionality, envy-freeness, Pareto optimality, and group strategyproofness.

- Computing this allocation is NP-hard.
  - Unlike DRF and matching under dichotomous preferences.

Leximin Strikes Again

- The result holds in a generic domain which satisfies:
  - Convexity: If two utility vectors are feasible, then so should be their convex combinations.
    - Holds if fractional or randomized allocations are allowed.
  - Equality: The maximum utility of each agent should be the same.
    - Normalize utilities.
  - Shifting Allocations: Swapping allocations of two agents should be allowed.
  - Maximal Utilization: No agent should have a higher utility for agent i's allocation than agent i has.
    - This should hold after the normalization. This is the most restrictive assumption.
- Captures DRF, matching with dichotomous preferences, classroom allocation, and many other settings from the literature.

Rent Division

Rent Division

- An apartment with n roommates & n rooms
- Roommates have preferences over the rooms
- Total rent is R

- Goal: Find an allocation of rooms to roommates & a division of the total rent that is envy-free.

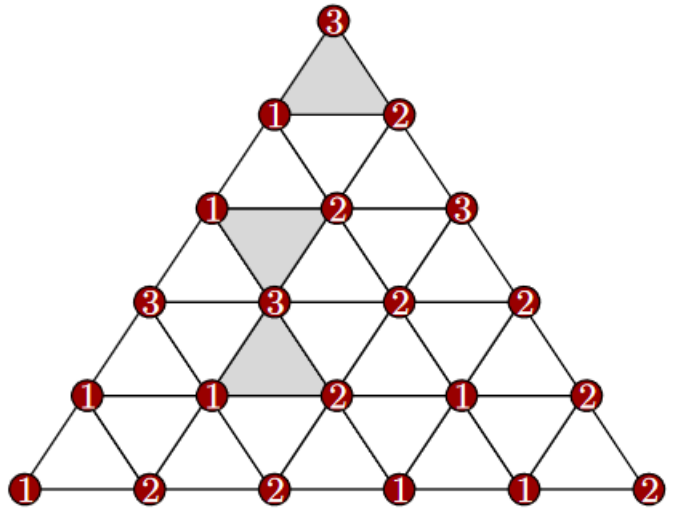
Model

- Allocation A
  - A_i = room given to i
- Rent division p
  - p_r = rent charged for room r, $\Sigma_r\, p_r = R$
- Utilities
  - $u_i(r, x)$ = utility to agent i for getting room r at rent x
- Envy-freeness:

$$\forall i, j : u_i\left(A_i, p_{A_i}\right) \geq u_i\left(A_j, p_{A_j}\right)$$

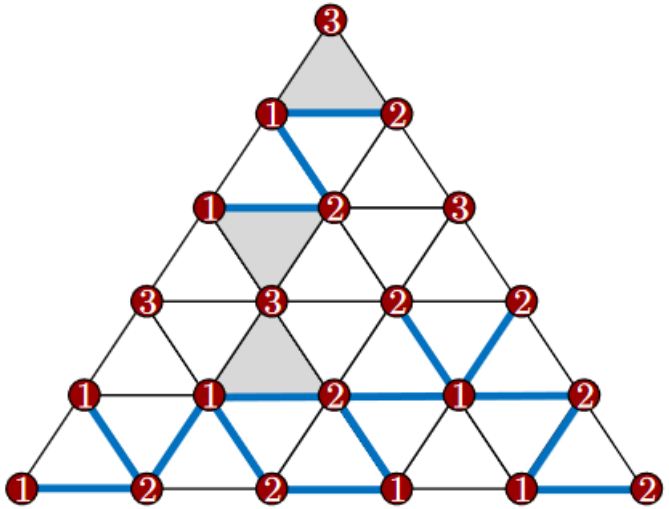
Sperner's Lemma

- Triangle T partitioned into elementary triangles
- Sperner Labeling:
  - Label vertices $\{1,2,3\}$
  - Main vertices are different
  - Vertices between main vertices i and j are each labeled i or j
- Lemma:
  - Any Sperner labeling contains at least one "fully labeled" (1-2-3) elementary triangle.



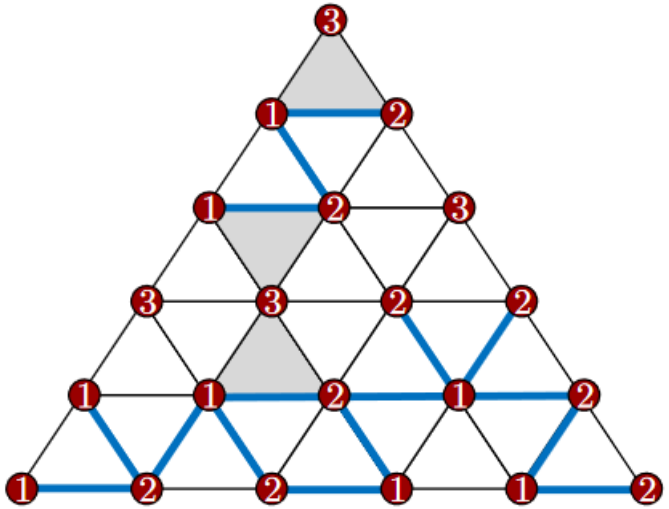
Sperner's Lemma

- Doors: 1-2 edges
- Rooms: elementary triangles
- Claim: #doors on the boundary of T is odd
- Claim: A fully labeled (123) room has 1 door. Every other room has 0 or 2 doors.



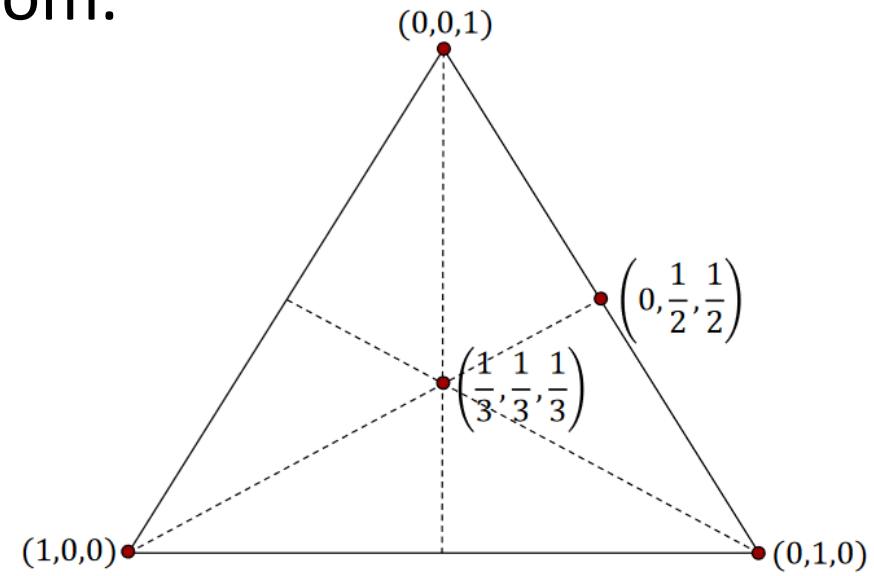
Sperner's Lemma

- Start at a door on boundary, and walk through it
- Either found a fully labeled room, or it has another door
- No room visited twice
- Eventually, find a fully labeled room or back out through another door on boundary
- But #doors on boundary is odd. ■



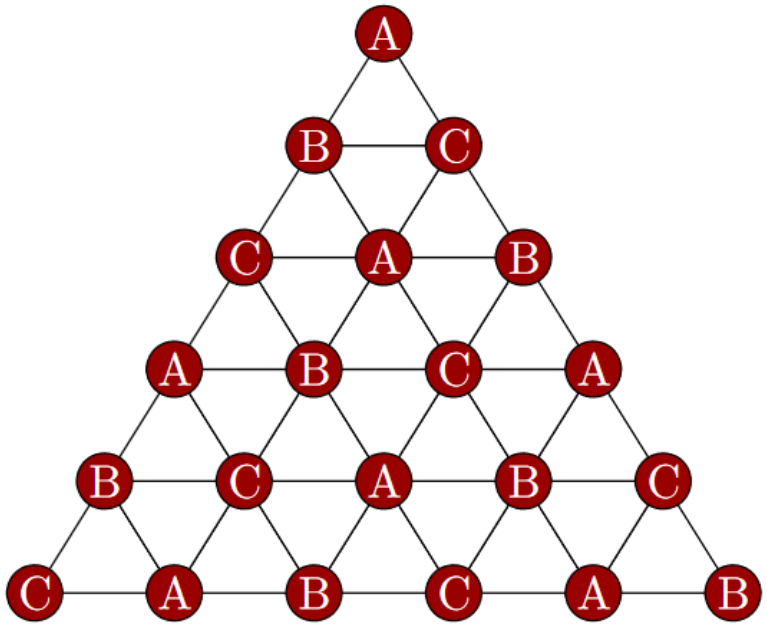
Fair Rent Division

- Three housemates A, B, C
- Goal: Divide total rent between three rooms so that at those rents, each person wants a different room.
- Without loss of generality, say the total rent is 1.
  - Represent possible partitions of rent as a triangle.



Fair Rent Division

- "Triangulate" and assign "ownership" of each vertex to A, B, or C so that each elementary triangle is an ABC triangle



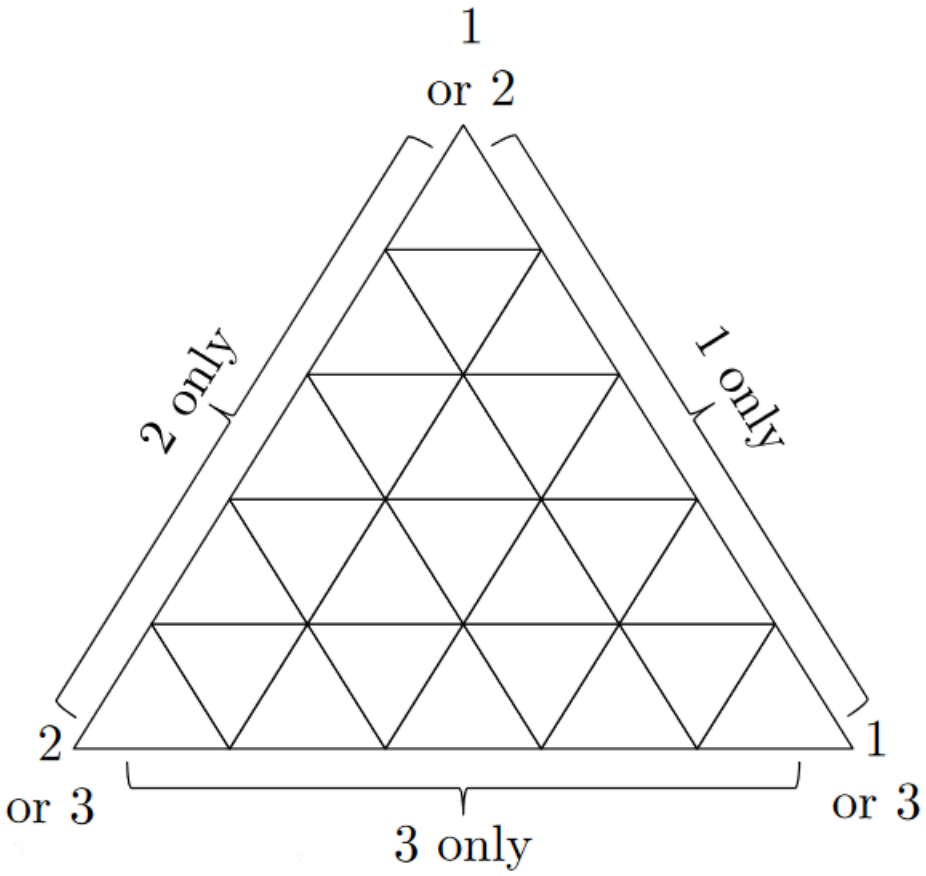
Fair Rent Division

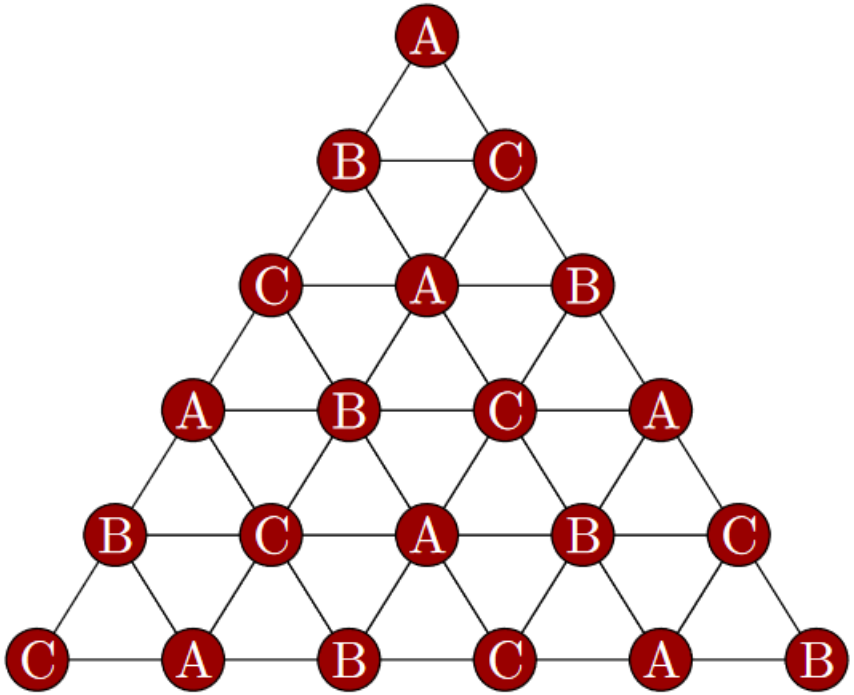
- Ask the owner of each vertex v:
  - Which room do you prefer if the rent division is given by the coordinates of v?

- Gives us a 1-2-3 labeling of the triangulation.
- Assumption: Each roommate prefers any free room over any paid room.
  - "Miserly roommates" assumption

Fair Rent Division

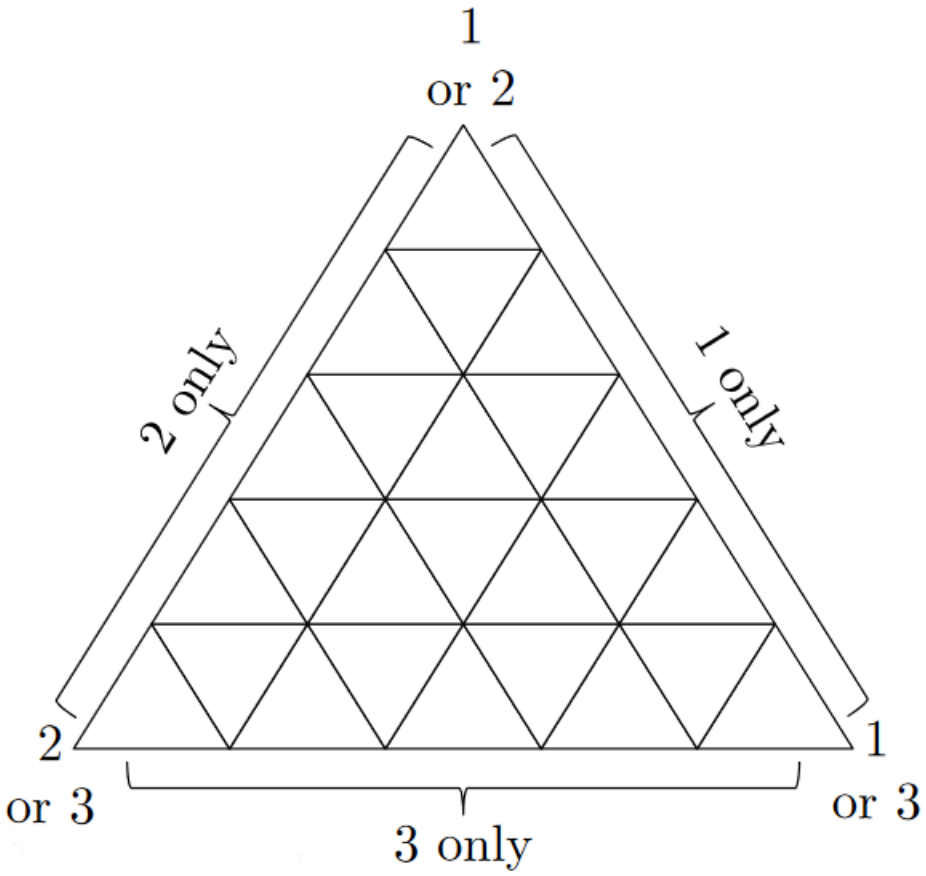
- This dictates the choice of rooms on the edges of T

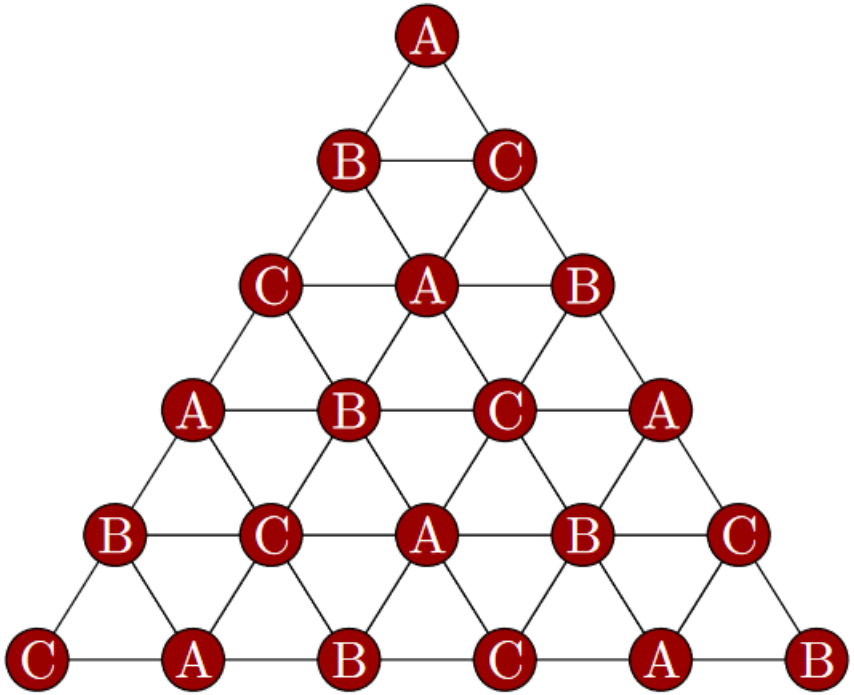




Fair Rent Division

- Sperner's Lemma: There must be a 1-2-3 triangle.





Fair Rent Division

- The three roommates prefer different rooms…
  - But at slightly different rent divisions.
  - Approximately envy-free.

- By making the triangulations finer, we can increase accuracy.
  - In the limit, we obtain an envy-free allocation.

- This technique generalizes to more roommates [Su 1999].

Quasi-Linear Utilities

- A special case of the general model, but different than assuming "miserly roommates"
  - Value of roommate i for room r = $v_{i,r}$
  - Rent for room r = p_r
  - Utility to agent i for getting room r = $v_{i,r} - p_r$

- We need to find an assignment A of rooms to roommates and a price vector p such that
  - Total rent: $R = \sum_r p_r$
  - Envy-freeness: $v_{i,A_i} - p_{A_i} \geq v_{i,A_j} - p_{A_j}$

Quasi-Linear Utilities

- Theorem: An envy-free (A, p) always exists!
  - We'll skip this proof.
- Theorem: If (A, p) is envy-free, $\sum_i v_{i,A_i}$ is maximized.
  - Implied by "1st fundamental theorem of welfare economics"
  - As a consequence, (A, p) is Pareto optimal.
  - Easy proof!
- Theorem: If (A, p) is envy-free and A' maximizes $\sum_i v_{i,A'_i}$ then (A', p) is envy-free.
  - Further, $v_{i,A_i} - p_{A_i} = v_{i,A'_i} - p_{A'_i}$ for every agent i
  - Implied by "2nd fundamental theorem of welfare economics"
  - Easy proof!

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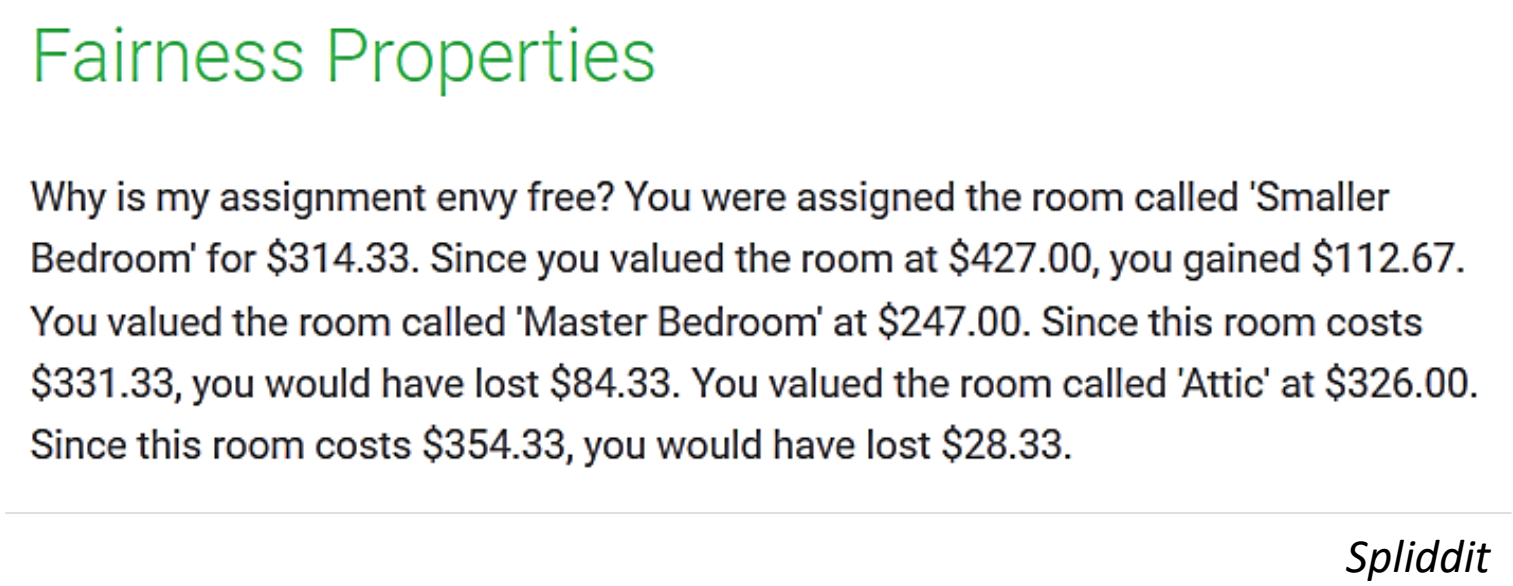
Suggest an App

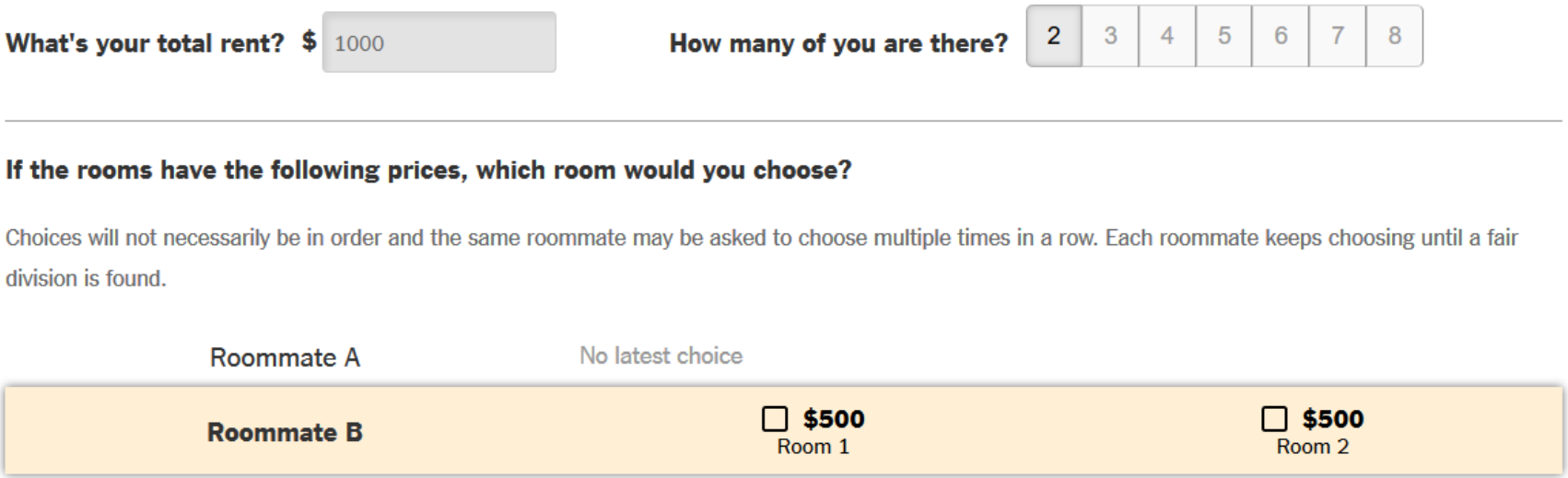
Which Model Is Better?

- Advantage of quasi-linear utilities:
  - One-shot preference elicitation
    - Players directly report their values for the different rooms
  - Easy to explain the fairness guarantee



Which Model Is Better?

- Advantage of miserly roommates model:
  - Allows arbitrary preferences subject to a simple assumption
  - Easy queries: "Which room do you prefer at these prices?"



The New York Times