Q1 VCG Mechanism for Auction
Suppose we want to sell three goods $a$, $b$, and $c$. There are three agents.

- Agent 1 either wants $\{a, b\}$ at $4$ or $\{c\}$ at $3$, or $\{a, b, c\}$ at $7$, and has zero value for other combinations.
- Agent 2 values $a$, $b$, and $c$ at $3$, $2$, and $2$, respectively, and has additive values (i.e., value of a set of goods = sum of values of goods in the set). So, for instance, agent 2’s value for $\{a, b\}$ would be $3 + 2 = 5$.
- Agent 3 values $a$, $b$, and $c$ at $2$, $3$, and $2$, respectively, and also has additive values.

Determine how the goods are allocated and how much do the three agents pay under VCG.

Q2 VCG for Public Policy
Suppose a government wants to choose a tax policy from three proposed alternatives $A$, $B$, and $C$. The government asks three lobbying groups ($L_1$, $L_2$, and $L_3$) to submit their values for the three proposals, and wants to run a VCG auction. If the following bids are submitted, which proposal would be selected, and how much would each group be charged?

\[
\begin{array}{c|ccc}
 & A & B & C \\
\hline
L_1 & 12 & 9 & 2 \\
L_2 & 5 & 8 & 10 \\
L_3 & 4 & 5 & 6 \\
\end{array}
\]

Q3 VCG for House Reassignment
In class, we argued that the VCG mechanism never pays agents; it only collects payments from agents. The proof was based on the assumption that the presence of an agent $i$ can only hurt the other agents. More formally, we assumed that the maximum possible welfare for agents other than $i$ when $i$ is absent is no less than the welfare they achieve under the VCG outcome when $i$ is present. This assumption holds in the settings of the two previous questions. Let us now look at a setting in which it is violated.

There are $n$ agents. Each agent owns a house. The agents are interested in possibly giving up their own house and moving to another agent’s house. The value that agent $i$ places on agent $j$’s house is $v_{i,j} \geq 0$ (thus, the value agent $i$ places on her own house is $v_{i,i}$). The space of outcomes $A$ is the set of all possible reassignments of the houses; formally, $A$ consists of all bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, where $\sigma(i)$ denotes the agent whose house is given to agent $i$. Note that $\sigma(i) = i$ is possible, in which case agent $i$ stays in her own house. Given an outcome $\sigma$, the
value derived by agent $i$ is $v_{i,\sigma(i)}$. Given payment $p_i$, agent $i$’s utility is $v_{i,\sigma(i)} - p_i$.

Crucially, note that when agent $i$ is absent, the space of outcomes is all bijections from $\{1, \ldots, n\} \setminus \{i\}$ to itself. That is, we are not allowed to use the house owned by agent $i$ and give it to someone else when agent $i$ is absent.

Construct an example in which the VCG mechanism assigns a negative payment to an agent (i.e., the agent is paid by the mechanism).