CSC304 Lecture 12

Mechanism Design w/ Money:
Revenue maximization
Myerson’s Auction
Revenue Maximization
Welfare vs Revenue

• In welfare maximization, we want to maximize $\sum_i v_i(a)$
  - VCG = strategyproof + maximizes welfare on every single instance
  - Beautiful!

• In revenue maximization, we want to maximize $\sum_i p_i$
  - We can still use strategyproof mechanisms (revelation principle).
  - BUT...
Welfare vs Revenue

• Different strategyproof mechanisms are better for different instances.

• Example: 1 item, 1 bidder, unknown value $v$
  - strategyproof = fix a price $r$, let the agent decide to “take it” ($v \geq r$) or “leave it” ($v < r$)
  - Maximize welfare $\rightarrow$ set $r = 0$
    - Must allocate item as long as the agent has a positive value
  - Maximize revenue $\rightarrow$ $r = ?$
    - Different $r$ are better for different $v$
Welfare vs Revenue

• We cannot optimize revenue on every instance
  ❆ Need to optimize the *expected* revenue in the Bayesian framework

• If we want to achieve higher expected revenue than VCG, we cannot always allocate the item
  ❆ Revenue equivalence principle!
Single Item + Single Bidder

• Value $v$ is drawn from distribution with CDF $F$
• **Goal:** post the optimal price $r$ on the item

• Revenue from price $r = r \cdot (1 - F(r))$ (Why?)

• **Optimal** $r^* = \text{argmax}_r \ r \cdot (1 - F(r))$
  ➢ “Monopoly price”
  ➢ Note: $r^*$ depends on $F$, but not on $v$, so the mechanism is strategyproof.
Example

• Suppose $F$ is the CDF of the uniform distribution over $[0, 1]$ (denote by $U[0, 1]$).
  ➢ CDF is given by $F(x) = x$ for all $x \in [0, 1]$.

• Recall: $E[\text{Revenue}]$ from price $r$ is $r \cdot (1 - F(r))$
  ➢ Q: What is the optimal posted price?
  ➢ Q: What is the corresponding optimal revenue?

• Compare this to the revenue of VCG, which is 0
  ➢ This is because if the value is less than $r^*$, we are willing to risk not selling the item.
Single Item + Two Bidders

• $v_1, v_2 \sim U[0,1]$

• VCG revenue = 2\textsuperscript{nd} highest bid = $\min(v_1, v_2)$
  - $E[\min(v_1, v_2)] = 1/3$ (Exercise!)

• Improvement: “VCG with reserve price”
  - Reserve price $r$
  - Highest bidder gets the item if bid more than $r$
  - Pays $\max(r, 2\textsuperscript{nd} \text{highest bid})$
    - “Critical payment” : Pay the least value you could have bid and still won the item
Single Item + Two Bidders

• Reserve prices are ubiquitous
  ➢ E.g., opening bids in eBay auctions
  ➢ Guarantee a certain revenue to auctioneer if item is sold

• $E[\text{revenue}] = E[\max(r, \min(v_1, v_2))]$
  ➢ Maximize over $r$? Hard to think about.

• What about a strategyproof mechanism that is not VCG + reserve price?
  ➢ What about just BNIC mechanisms?
Single-Parameter Environments

• Roger B. Myerson solved revenue optimal auctions in “single-parameter environments”

• Proposed a simple auction that maximizes expected revenue
Single-Parameter Environments

• Each agent $i$...
  ➢ has a private value $v_i$ drawn from a distribution with CDF $F_i$ and PDF $f_i$
  ➢ is “satisfied” at some level $x_i \in [0,1]$, which gives the agent value $x_i \cdot v_i$
  ➢ is asked to pay $p_i$

• Examples
  ➢ Single divisible item
  ➢ Single indivisible item ($x_i \in \{0,1\}$ – this is okay too!)
  ➢ Many items, single-minded bidders (again $x_i \in \{0,1\}$)
Myerson’s Lemma

- **Myerson’s Lemma:**
  For a single-parameter environment, a mechanism is strategyproof if and only if for all $i$
  
  1. $x_i$ is monotone non-decreasing in $v_i$
  2. $p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$

(typically, $p_i(0) = 0$)

- Generalizes critical payments
  - For every “$\delta$” allocation, pay the lowest value that would have won it
Myerson’s Lemma

• Note: allocation determines unique payments

\[ p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0) \]

• A corollary: revenue equivalence
  ➢ If two mechanisms use the same allocation \( x_i \), they “essentially” have the same expected revenue

• Another corollary: optimal revenue auctions
  ➢ Optimizing revenue = optimizing some function of allocation (easier to analyze)
Myerson’s Theorem

• “Expected Revenue = Expected Virtual Welfare”
  ➢ Recall: \( p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0) \)
  ➢ Take expectation over draw of valuations + lots of calculus

\[
E_{\{v_i \sim F_i\}}[\sum_i p_i] = E_{\{v_i \sim F_i\}}[\sum_i \varphi_i \cdot x_i]
\]

• \( \varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \) = virtual value of bidder \( i \)

• \( \sum_i \varphi_i \cdot x_i = \text{virtual welfare} \)
Myerson’s Theorem

• **Myerson’s auction:**
  - A strategyproof auction maximizes the (expected) revenue if its allocation rule maximizes the virtual welfare subject to monotonicity and it charges critical payments.

• Charging critical payments is easy.

• But maximizing virtual welfare *subject to monotonicity* is tricky.
  - Let’s get rid of the monotonicity requirement!
Myerson’s Theorem Simplified

• Regular Distributions
  ➢ A distribution $F$ is regular if its virtual value function $\varphi(v) = v - (1 - F(v))/f(v)$ is non-decreasing in $v$.
  ➢ Many important distributions are regular, e.g., uniform, exponential, Gaussian, power-law, ...

• Lemma
  ➢ If all $F_i$’s are regular, the allocation rule maximizing virtual welfare is already monotone.

• Myerson’s Corollary:
  ➢ When all $F_i$’s are regular, the strategyproof auction maximizes virtual welfare and charges critical payments.
Single Item + Single Bidder

• Setup:
  ➢ Single indivisible item, single bidder, value $v$ drawn from a regular distribution with CDF $F$ and PDF $f$

• Goal:
  ➢ Maximize $\varphi \cdot x$, where $\varphi = v - \frac{1-F(v)}{f(v)}$ and $x \in \{0,1\}$

• Optimal auction:
  ➢ $x = 1$ iff $\varphi \geq 0 \iff v \geq \frac{1-F(v)}{f(v)} \iff v \geq v^*$ where $v^* = \frac{1-F(v^*)}{f(v^*)}$
  ➢ Critical payment: $v^*$
  ➢ This is VCG with a reserve price of $\varphi^{-1}(0)$!
Example

• Optimal auction:
  ➢ $x = 1$ iff $\varphi \geq 0 \iff v \geq \frac{1-F(v)}{f(v)}$
  ➢ Critical payment: $v^*$ such that $v^* = \frac{1-F(v^*)}{f(v^*)}$

• Distribution is $U[0,1]$:  
  ➢ $x = 1$ iff $v \geq \frac{1-v}{1} \iff v \geq \frac{1}{2}$
  ➢ Critical payment = $\frac{1}{2}$
  ➢ That is, we post the optimal price of 0.5
Single Item + $n$ Bidders

• Setup:
  ➢ Single indivisible item, each bidder $i$ has value $v_i$ drawn from a regular distribution with CDF $F_i$ and PDF $f_i$

• Goal:
  ➢ Maximize $\sum_i \varphi_i \cdot x_i$ where $\varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $x_i \in \{0,1\}$ such that $\sum_i x_i \leq 1$
Single Item + n Bidders

• Optimal auction:

  ➢ If all $\varphi_i < 0$:
    o Nobody gets the item, nobody pays anything
    o For all $i$, $x_i = p_i = 0$
  
  ➢ If some $\varphi_i \geq 0$:
    o Agent with highest $\varphi_i$ wins the item, pays critical payment
    o $i^* \in \text{argmax}_i \varphi_i(v_i), x_{i^*} = 1, x_i = 0 \ \forall i \neq i^*$
    o $p_{i^*} = \varphi_{i^*}^{-1}\left(\max\left(0, \max_{j \neq i^*} \varphi_j(v_j)\right)\right)$

• Note: The item doesn’t necessarily go to the highest value agent!
Special Case: iid Values

- Suppose all distributions are identical (say CDF $F$ and PDF $f$)

- Check that the auction simplifies to the following
  - Allocation: item goes to bidder $i^*$ with highest value if her value $v_{i^*} \geq \phi^{-1}(0)$
  - Payment charged = $\max(\phi^{-1}(0), \max_{j \neq i^*} v_j)$

- This is again VCG with a reserve price of $\phi^{-1}(0)$
Example

- Two bidders, both drawing iid values from $U[0,1]$
  - $\varphi(v) = v - \frac{1-v}{1} = 2v - 1$
  - $\varphi^{-1}(0) = 1/2$

- Auction:
  - Give the item to the highest bidder if their value is at least $1/2$
  - Charge them $\max(\frac{1}{2}, 2^{\text{nd}} \text{ highest bid})$
Example

• Two bidders, one with value from \( U[0,1] \), one with value from \( U[3,5] \)

  \[
  \varphi_1(v_1) = 2v_1 - 1 \\
  \varphi_2(v_2) = v_2 - \frac{1-F_2(v_2)}{f_2(v_2)} = v_2 - \frac{1-v_2-\frac{3}{2}}{1/2} = 2v_2 - 5
  \]

• Auction:

  \[
  \text{If } v_1 < \frac{1}{2} \text{ and } v_2 < \frac{5}{2}, \text{ the item remains unallocated.}
  \]

  \[
  \text{Otherwise...}
  \]

  \[
  \text{o If } 2v_1 - 1 > 2v_2 - 5, \text{ agent 1 gets it and pays max}(\frac{1}{2}, v_2 - 2) \\
  \text{o If } 2v_1 - 1 < 2v_2 - 5, \text{ agent 2 gets it and pays max}(\frac{5}{2}, v_1 + 2)
  \]
Extensions

• Irregular distributions:
  ➢ E.g., multi-modal or extremely heavy tail distributions
  ➢ Need to add the monotonicity constraint
  ➢ Turns out, we can “iron” irregular distributions to make them regular and then use Myerson’s framework

• Relaxing DSIC to BNIC
  ➢ Myerson’s mechanism has optimal revenue among all DSIC mechanisms
  ➢ Turns out, it also has optimal revenue among the much larger class of BNIC mechanisms!
Approx. Optimal Auctions

• Optimal auctions become unintuitive and difficult to understand with unequal distributions, even if they are regular
  ➢ Simpler auctions preferred in practice
  ➢ We still want approximately optimal revenue

• Theorem [Hartline & Roughgarden, 2009]:
  ➢ For iid values from regular distributions, VCG with bidder-specific reserve prices gives a 2-approximation of the optimal revenue.
Approximately Optimal

• Still relies on knowing bidders’ distributions

• Theorem [Bulow and Klemperer, 1996]:
  
  For i.i.d. values, 
  
  $E[\text{Revenue of VCG with } n + 1 \text{ bidders}] \geq E[\text{Optimal revenue with } n \text{ bidders}]$

• “Spend that effort in getting one more bidder than in figuring out the optimal auction”
Simple proof

• One can show that VCG with \( n + 1 \) bidders has the max revenue among all \( n + 1 \) bidder strategyproof auctions that always allocate the item via revenue equivalence.

• Consider the auction: “Run \( n \)-bidder Myerson on the first \( n \) bidders. If the item is unallocated, give it to agent \( n + 1 \) for free.”
  - \( n + 1 \) bidder DSIC auction
  - As much revenue as \( n \)-bidder Myerson auction
Optimizing Revenue is Hard

• Slow progress beyond single-parameter setting
  ➢ Even with just two items and one bidder with i.i.d. values for both items, the optimal auction DOES NOT run Myerson’s auction on individual items!
  ➢ “Take-it-or-leave-it” offers for the two items bundled might increase revenue

• But nowadays, the focus is on simple, approximately optimal auctions instead of complicated, optimal auctions.