CSC304 Lecture 21

Fair Division 2:
Cake-cutting, Indivisible goods

Recall: Cake-Cutting

- A heterogeneous, divisible good
  - Represented as $[0,1]$

- Set of players $N = \{1, \ldots, n\}$
  - Each player i has valuation V_i
- Allocation $A = (A_1, \ldots, A_n)$
  - Disjoint partition of the cake

Recall: Cake-Cutting

- We looked at two measures of fairness:

- Proportionality: $\forall i \in N: V_i(A_i) \geq 1/n$
  - "Every agent should get her fair share."

- Envy-freeness: $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$
  - "No agent should prefer someone else's allocation."

Four More Desiderata

- Equitability
  - $V_i(A_i) = V_j(A_j)$ for all i, j.
- Perfect Partition
  - $V_i(A_k) = 1/n$ for all i, k.
  - Implies equitability.
  - Guaranteed to exist [Lyapunov '40] and can be found using only poly(n) cuts [Alon '87].

Four More Desiderata

- Pareto Optimality
  - We say that A is Pareto optimal if for any other allocation B, it cannot be that $V_i(B_i) \geq V_i(A_i)$ for all i and $V_i(B_i) > V_i(A_i)$ for some i.

- Strategyproofness
  - No agent can misreport her valuation and increase her (expected) value for her allocation.

Strategyproofness

- Deterministic
  - Bad news!
  - Theorem [Menon & Larson '17]: No deterministic SP mechanism is (even approximately) proportional.
- Randomized
  - Good news!
  - Theorem [Chen et al. '13, Mossel & Tamuz '10]: There is a randomized SP mechanism that *always* returns an envy-free allocation.

Strategyproofness

- Randomized SP Mechanism:
  - Compute a perfect partition, and assign the n bundles to the n players uniformly at random.
- Why is this EF?
  - Every agent has value $1/n$ for her own as well as for every other agent's allocation.
  - Note: We want EF in every realized allocation, not only in expectation.
- Why is this SP?
  - An agent is assigned a random bundle, so her expected utility is $1/n$, irrespective of what she reports.

Pareto Optimality (PO)

- Definition: We say that A is Pareto optimal if for any other allocation B, it cannot be that $V_i(B_i) \geq V_i(A_i)$ for all i and $V_i(B_i) > V_i(A_i)$ for some i.

- Q: Is it PO to give the entire cake to player 1?
- A: Not necessarily. But yes if player 1 values "every part of the cake positively".

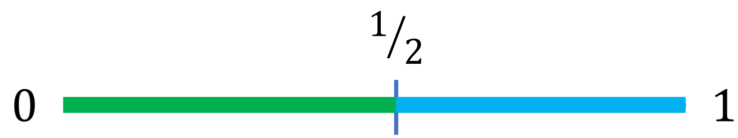
PO + EF

- Theorem [Weller '85]:
  - There always exists an allocation of the cake that is both envy-free and Pareto optimal.
- One way to achieve PO+EF:
  - Nash-optimal allocation: $\text{argmax}_A \prod_{i \in N} V_i(A_i)$
  - Obviously, this is PO. The fact that it is EF is non-trivial.
  - This is named after John Nash.
    - Nash social welfare = product of utilities
    - Different from utilitarian social welfare = sum of utilities

Nash-Optimal Allocation

- Example:
  - Green player has value 1 distributed evenly over $[0, {}^2/_3]$
  - Blue player has value 1 distributed evenly over $[0,1]$
  - Without loss of generality (why?) suppose:
    - Green player gets $[0, x]$ for $x \leq {}^2/_3$
    - Blue player gets $[x, {}^2/_3] \cup [{}^2/_3, 1] = [x, 1]$
  - Green's utility = $\frac{x}{{}^2/_3}$, blue's utility = $1 - x$
  - Maximize: $\frac{3}{2} x \cdot (1 - x) \Rightarrow x = {}^1/_2$

Allocation



Green has utility $\frac{3}{4}$

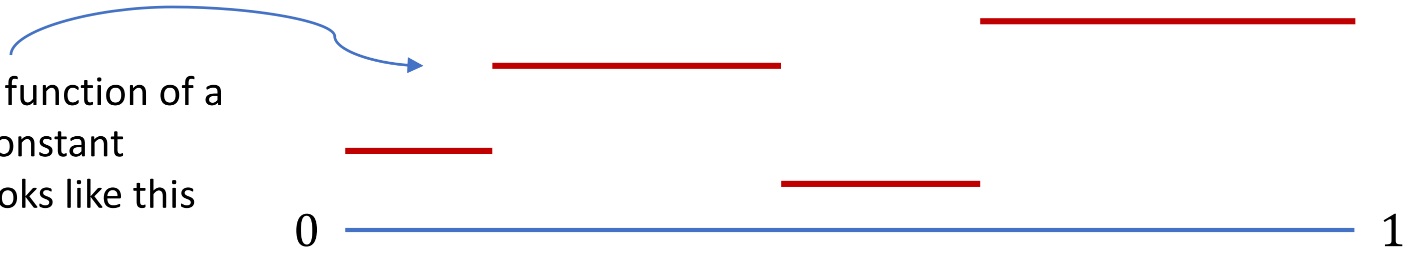
Blue has utility $\frac{1}{2}$

Problem

- Difficult to compute in general
  - I believe it should require an unbounded number of queries in the Robertson-Webb model. But I can't find such a result in the literature.

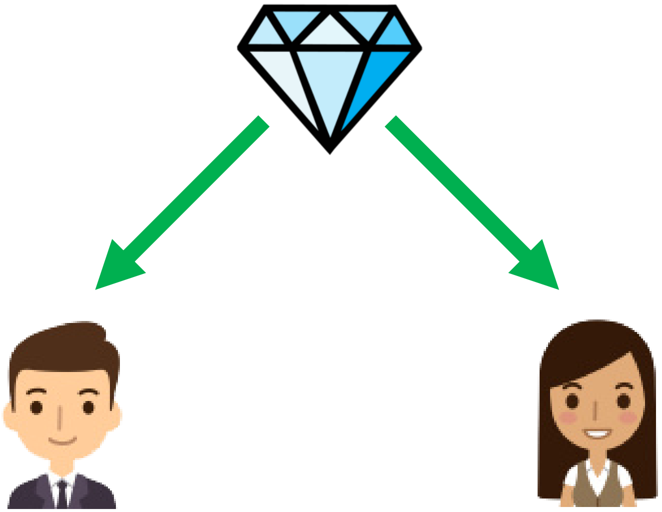
- Theorem [Aziz & Ye '14]:
  - For *piecewise constant* valuations, the Nash-optimal solution can be computed in polynomial time.

The density function of a piecewise constant valuation looks like this



Indivisible Goods

- Goods cannot be shared / divided among players
  - E.g., house, painting, car, jewelry, …
- Problem: Envy-free allocations may not exist!



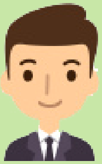
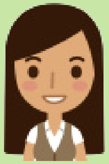
Indivisible Goods: Setting

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Given such a matrix of numbers, assign each good to a player.

We assume additive values. So, e.g., V$(\{$$,$$\}) = 8 + 7 = 15$

Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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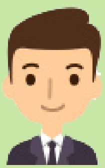
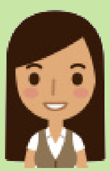
Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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Indivisible Goods

				
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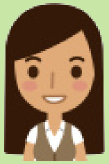	9	11	12	8
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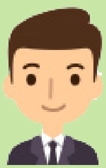
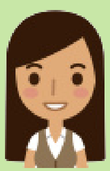
Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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Indivisible Goods

- Envy-freeness up to one good (EF1):

$$\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \backslash \{g\}\right)$$

  - Technically, $\exists g \in A_j$ only applied if $A_j \neq \emptyset$.
  - "If i envies j, there must be some good in j's bundle such that removing it would make i envy-free of j."

- Does there always exist an EF1 allocation?

EF1

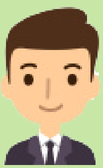
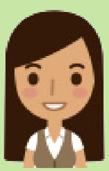
- Yes! We can use Round Robin.
  - Agents take turns in a cyclic order, say $1,2, \ldots, n, 1,2, \ldots, n, \ldots$
  - An agent, in her turn, picks the good that she likes the most among the goods still not picked by anyone.
  - [Assignment Problem] This yields an EF1 allocation regardless of how you order the agents.

- Sadly, the allocation returned may not be Pareto optimal.

EF1+PO?

- Nash welfare to the rescue!
- Theorem [Caragiannis et al. '16]:
  - Maximizing Nash welfare achieves both EF1 and PO.
  - But what if there are two goods and three players?
    - All allocations have zero Nash welfare (product of utilities).
    - But we cannot give both goods to a single player.
  - Algorithm in detail:
    - Step 1: Choose a subset of players $S \subseteq N$ with the largest $|S|$ such that it is possible to give every player in S positive utility simultaneously.
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$

Integral Nash Allocation

				
	8	7	20	5
	9	11	12	8
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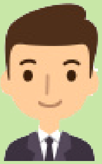
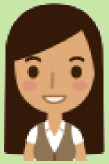
$$20 * 8 * (9+10) = 3040$$

				
	8	7	20	5
	9	11	12	8
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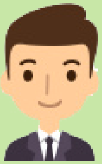
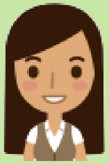
(8+7) * 8 * 18 = 2160

				
	8	7	20	5
	9	11	12	8
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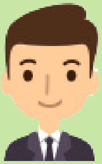
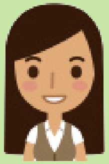
8 * (12+8) * 10 = 1600

				
	8	7	20	5
	9	11	12	8
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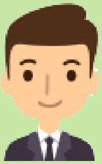
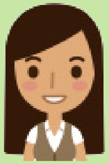
$$20 * (11+8) * 9 = 3420$$

				
	8	7	20	5
	9	11	12	8
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Computation

- For indivisible goods, Nash-optimal solution is strongly NP-hard to compute
  - That is, remains NP-hard even if all values are bounded.

- Open Question: Can we find an allocation that is both EF1 and PO in polynomial time?
  - A recent paper provides a pseudo-polynomial time algorithm, i.e., its time is polynomial in n, m, and $\max_{i,g} V_i(\{g\})$.

Stronger Fairness Guarantees

- Envy-freeness up to the least valued good (EFx):
  - $\forall i, j \in N, \forall g \in A_j : V_i(A_i) \geq V_i(A_j \setminus \{g\})$
  - "If i envies j, then removing *any* good from j's bundle eliminates the envy."
  - Open question: Is there always an EFx allocation?

- Contrast this with EF1:
  - $\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i(A_j \setminus \{g\})$
  - "If i envies j, then removing *some* good from j's bundle eliminates the envy."
  - We know there is always an EF1 allocation that is also PO.

Stronger Fairness

- To clarify the difference between EF1 and EFx:
  - Suppose there are two players and three goods with values as follows.

	A	B	C
P1	5	1	10
P2	0	1	10

  - If you give {A} → P1 and {B,C} → P2, it's EF1 but not EFx.
    - EF1 because if P1 removes C from P2's bundle, all is fine.
    - Not EFx because removing B doesn't eliminate envy.
  - Instead, {A,B} → P1 and {C} → P2 would be EFx.