CSC304 Lectures 4 & 5

Game Theory
(PoA, PoS, Cost sharing & congestion games, Potential function, Braess’ paradox)
Recap

• Nash equilibria (NE)
  ➢ No agent wants to change their strategy
  ➢ Guaranteed to exist if mixed strategies are allowed
  ➢ Could be multiple

• Pure NE through best-response diagrams
• Mixed NE through the indifference principle
Worst and Best Nash Equilibria

• What can we say after we identify all Nash equilibria?
  ➢ Compute how “good” they are in the best/worst case

• How do we measure “social good”?
  ➢ Game with only rewards?
    Higher total reward of players = more social good
  ➢ Game with only penalties?
    Lower total penalty to players = more social good
  ➢ Game with rewards and penalties?
    No clear consensus...
Price of Anarchy and Stability

• Price of Anarchy (PoA)
  “Worst NE vs optimum”
  \[ \frac{\text{Max total reward}}{\text{Min total reward in any NE}} \]

  or

  \[ \frac{\text{Max total cost in any NE}}{\text{Min total cost}} \]

• Price of Stability (PoS)
  “Best NE vs optimum”
  \[ \frac{\text{Max total reward}}{\text{Max total reward in any NE}} \]

  or

  \[ \frac{\text{Min total cost in any NE}}{\text{Min total cost}} \]

\[ \text{PoA} \geq \text{PoS} \geq 1 \]
Revisiting Stag-Hunt

Max total reward = 4 + 4 = 8

Three equilibria

- (Stag, Stag) : Total reward = 8
- (Hare, Hare) : Total reward = 2
- (\(\frac{1}{3}\) Stag – \(\frac{2}{3}\) Hare, \(\frac{1}{3}\) Stag – \(\frac{2}{3}\) Hare)
  - Total reward = \(\frac{1}{3} \times \frac{1}{3} \times 8 + \left(1 - \frac{1}{3} \times \frac{1}{3}\right) \times 2 \in (2,8)\)

Price of stability? Price of anarchy?
Revisiting Prisoner’s Dilemma

<table>
<thead>
<tr>
<th>Sam</th>
<th>John</th>
<th>Stay Silent</th>
<th>Betray</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stay Silent</td>
<td>(-1, -1)</td>
<td>(-3, 0)</td>
<td></td>
</tr>
<tr>
<td>Betray</td>
<td>(0, -3)</td>
<td>(-2, -2)</td>
<td></td>
</tr>
</tbody>
</table>

- Min total cost = 1 + 1 = 2
- Only equilibrium:
  - (Betray, Betray) : Total cost = 2 + 2 = 4
- Price of stability? Price of anarchy?
Cost Sharing Game

• $n$ players on directed weighted graph $G$

• Player $i$
  - Wants to go from $s_i$ to $t_i$
  - Strategy set $S_i = \{\text{directed } s_i \rightarrow t_i \text{ paths}\}$
  - Denote his chosen path by $P_i \in S_i$

• Each edge $e$ has cost $c_e$ (weight)
  - Cost is split among all players taking edge $e$
  - That is, among all players $i$ with $e \in P_i$
Cost Sharing Game

• Given strategy profile \( \vec{P} \), cost \( c_i(\vec{P}) \) to player \( i \) is sum of his costs for edges \( e \in P_i \)

• Social cost \( C(\vec{P}) = \sum_i c_i(\vec{P}) \)

• Note: \( C(\vec{P}) = \sum_{e \in E(\vec{P})} c_e \), where...
  ➢ \( E(\vec{P}) = \{ \text{edges taken in } \vec{P} \text{ by at least one player} \} \)
  ➢ Why?
Cost Sharing Game

• In the example on the right:
  ➢ What if both players take direct paths?
  ➢ What if both take middle paths?
  ➢ What if one player takes direct path and the other takes middle path?

• Pure Nash equilibria?
Cost Sharing: Simple Example

• Example on the right: $n$ players

• Two pure NE
  - All taking the $n$-edge: social cost = $n$
  - All taking the 1-edge: social cost = 1
    o Also the social optimum

• Price of stability: 1

• Price of anarchy: $n$
  - We can show that price of anarchy $\leq n$ in every cost-sharing game!
Cost Sharing: PoA

• **Theorem:** The price of anarchy of a cost sharing game is at most $n$.

• **Proof:**
  - Suppose the social optimum is $(P_1^*, P_2^*, ..., P_n^*)$, in which the cost to player $i$ is $c_i^*$.
  - Take any NE with cost $c_i$ to player $i$.
  - Let $c_i'$ be his cost if he switches to $P_i^*$.
  - NE $\Rightarrow c_i' \geq c_i$  (Why?)
  - But : $c_i' \leq n \cdot c_i^*$  (Why?)
  - $c_i \leq n \cdot c_i^*$ for each $i$ $\Rightarrow$ no worse than $n \times$ optimum
Cost Sharing

• Price of anarchy
  ➢ Every cost-sharing game: PoA ≤ n
  ➢ Example game with PoA = n
  ➢ Bound of n is tight.

• Price of stability?
  ➢ In the previous game, it was 1.
  ➢ In general, it can be higher. How high?
  ➢ We’ll answer this after a short detour.
Cost Sharing

• Nash’s theorem shows existence of a mixed NE.
  ➢ Pure NE may not always exist in general.

• But in both cost-sharing games we saw, there was a PNE.
  ➢ What about a more complex game like the one on the right?

10 players: $E \rightarrow C$
27 players: $B \rightarrow D$
19 players: $C \rightarrow D$
Good News

- **Theorem:** Every cost-sharing game have a pure Nash equilibrium.

- **Proof:**
  - Via “potential function” argument
Step 1: Define Potential Fn

- **Potential function**: \( \Phi : \prod_i S_i \rightarrow \mathbb{R}_+ \)
  
  - This is a function such that for every pure strategy profile \( \vec{P} = (P_1, ..., P_n) \), player \( i \), and strategy \( P_i' \) of \( i \),
    
    \[
    c_i \left( P_i', \vec{P}_{-i} \right) - c_i \left( \vec{P} \right) = \Phi \left( P_i', \vec{P}_{-i} \right) - \Phi \left( \vec{P} \right)
    \]

  - When a single player \( i \) changes her strategy, the change in potential function **equals the change in cost to \( i \)**!

- **Note**: In contrast, the change in the social cost \( C \) equals the total change in cost to all players.
  
  - Hence, the social cost will often not be a valid potential function.
Step 2: Potential $F^n \rightarrow$ pure Nash Eq

• A potential function exists $\Rightarrow$ a pure NE exists.
  ➢ Consider a $\vec{P}$ that minimizes the potential function.
  ➢ Deviation by any single player $i$ can only (weakly) increase the potential function.
  ➢ But change in potential function = change in cost to $i$.
  ➢ Hence, there is no beneficial deviation for any player.

• Hence, every pure strategy profile minimizing the potential function is a pure Nash equilibrium.
Step 3: Potential $F^n$ for Cost-Sharing

• Recall: $E(\tilde{P}) = \{\text{edges taken in } \tilde{P} \text{ by at least one player}\}$

• Let $n_e(\tilde{P})$ be the number of players taking $e$ in $\tilde{P}$

$$\Phi(\tilde{P}) = \sum_{e \in E(\tilde{P})} \sum_{k=1}^{n_e(\tilde{P})} \frac{c_e}{k}$$

• Note: The cost of edge $e$ to each player taking $e$ is $c_e/n_e(\tilde{P})$. But the potential function includes all fractions: $c_e/1, c_e/2, \ldots, c_e/n_e(\tilde{P})$. 
Step 3: Potential $F^n$ for Cost-Sharing

$$\Phi (\vec{P}) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k}$$

• Why is this a potential function?
  ➢ If a player changes path, he pays $\frac{c_e}{n_e(\vec{P})+1}$ for each new edge $e$, gets back $\frac{c_f}{n_f(\vec{P})}$ for each old edge $f$.
  ➢ This is precisely the change in the potential function too.
  ➢ So $\Delta c_i = \Delta \Phi$. 

∎
Potential Minimizing Eq.

• Minimizing the potential function gives some pure Nash equilibrium
  ➢ Is this equilibrium special? Yes!

• Recall that the price of anarchy can be up to $n$.
  ➢ That is, the worst Nash equilibrium can be up to $n$ times worse than the social optimum.

• A potential-minimizing pure Nash equilibrium is better!
Potential Minimizing Eq.

\[ \sum_{e \in E(\vec{P})} c_e \leq \Phi(\vec{P}) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k} \leq \sum_{e \in E(\vec{P})} c_e \ast \sum_{k=1}^{n} \frac{1}{k} \]

Social cost

\[ \forall \vec{P}, \quad C(\vec{P}) \leq \Phi(\vec{P}) \leq C(\vec{P}) \ast H(n) \]

Harmonic function \( H(n) = \sum_{k=1}^{n} 1/n = O(\log n) \)

Potential minimizing eq.

Social optimum
Potential Minimizing Eq.

• Potential-minimizing PNE is $O(\log n)$-approximation to the social optimum.

• Thus, in every cost-sharing game, the price of stability is $O(\log n)$.
  - Compare to the price of anarchy, which can be $n$
Congestion Games

• Generalize cost sharing games

• $n$ players, $m$ resources (e.g., edges)

• Each player $i$ chooses a set of resources $P_i$ (e.g., $s_i \rightarrow t_i$ paths)

• When $n_j$ player use resource $j$, each of them get a cost $f_j(n_j)$

• Cost to player is the sum of costs of resources used
Congestion Games

- **Theorem [Rosenthal 1973]:** Every congestion game is a potential game.

- Potential function:

\[ \Phi(\vec{P}) = \sum_{j \in E(\vec{P})} \sum_{k=1}^{n_j(\vec{P})} f_j(k) \]

- **Theorem [Monderer and Shapley 1996]:** Every potential game is equivalent to a congestion game.
Potential Functions

- Potential functions are useful for deriving various results
  - E.g., used for analyzing amortized complexity of algorithms

- **Bad news**: Finding a potential function that works may be hard.
The Braess’ Paradox

• In cost sharing, $f_j$ is decreasing
  ➢ The more people use a resource, the less the cost to each.

• $f_j$ can also be increasing
  ➢ Road network, each player going from home to work
  ➢ Uses a sequence of roads
  ➢ The more people on a road, the greater the congestion, the greater the delay (cost)

• Can lead to **unintuitive phenomena**
The Braess’ Paradox

- Parkes-Seuken Example
  - 2000 players want to go from 1 to 4
  - 1 → 2 and 3 → 4 are “congestible” roads
  - 1 → 3 and 2 → 4 are “constant delay” roads

\[
c_{12}(n_{12}) = \frac{n_{12}}{100} \quad c_{24}(n_{24}) = 25
\]

\[
c_{13}(n_{13}) = 25 \quad c_{34}(n_{34}) = \frac{n_{34}}{100}
\]
The Braess’ Paradox

• Pure Nash equilibrium?
  ➢ 1000 take 1 → 2 → 4, 1000 take 1 → 3 → 4
  ➢ Each player has cost $10 + 25 = 35$
  ➢ Anyone switching to the other creates a greater congestion on it, and faces a higher cost

![Diagram of network flow with costs and routes]
The Braess’ Paradox

• What if we add a zero-cost connection 2 → 3?
  - Intuitively, adding more roads should only be helpful
  - In reality, it leads to a greater delay for everyone in the unique equilibrium!

![Diagram of network with nodes and costs]

- $c_{12}(n_{12}) = \frac{n_{12}}{100}$
- $c_{23}(n_{23}) = 0$
- $c_{24}(n_{24}) = 25$
- $c_{13}(n_{13}) = 25$
- $c_{34}(n_{34}) = \frac{n_{34}}{100}$
The Braess’ Paradox

• Nobody chooses 1 → 3 as 1 → 2 → 3 is better irrespective of how many other players take it
• Similarly, nobody chooses 2 → 4
• Everyone takes 1 → 2 → 3 → 4, faces delay = 40!

\[
\begin{align*}
    c_{12}(n_{12}) &= \frac{n_{12}}{100} \\
    c_{24}(n_{24}) &= 25 \\
    c_{23}(n_{23}) &= 0 \\
    c_{13}(n_{13}) &= 25 \\
    c_{34}(n_{34}) &= \frac{n_{34}}{100}
\end{align*}
\]
The Braess’ Paradox

• In fact, what we showed is:

➢ In the new game, $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ is a strictly dominant strategy for each player!

\[ c_{12}(n_{12}) = \frac{n_{12}}{100} \quad c_{24}(n_{24}) = 25 \]

\[ c_{23}(n_{23}) = 0 \quad c_{13}(n_{13}) = 25 \]

\[ c_{34}(n_{34}) = \frac{n_{34}}{100} \]