CSC304 Lecture 9

Mechanism Design with Money:
More VCG examples;
greedy approximation of VCG
VCG Recap

- \( f(\tilde{\nu}) = a^* = \arg\max_{a \in A} \sum_i \tilde{\nu}_i(a) \)
  - Choose the allocation maximizing reported welfare

- \( p_i(\tilde{\nu}) = \left[ \max_a \sum_{j \neq i} \tilde{\nu}_j(a) \right] - \left[ \sum_{j \neq i} \tilde{\nu}_j(a^*) \right] \)
  - Each agent pays the loss to others due to her presence

- Four properties
  - Strategyproofness
  - Individual rationality (IR)
  - No payments to agents
  - Welfare maximization
Seller as Agent

• Seller ($S$) wants to sell his car ($c$) to buyer ($B$)

• Seller has a value for his own car: $v_S(c)$
  - Individual rationality for the seller mandates that seller must get revenue at least $v_S(c)$

• Idea: Add seller as another agent, and make his values part of the welfare calculations!
Seller as Agent

\[ v_S(c) = 3 \quad \text{and} \quad v_B(c) = 5 \]

• What if...
  - We give the car to buyer when \( v_B(c) > v_S(c) \) and
  - Buyer pays seller \( v_B(c) \): Not strategyproof for buyer!
  - Buyer pays seller \( v_S(c) \): Not strategyproof for seller!
What would VCG do?

\[ v_S(c) = 3 \quad \text{and} \quad v_B(c) = 5 \]

- **Allocation?**
  - Buyer gets the car (welfare = 5)

- **Payment?**
  - Buyer pays: \( 3 - 0 = 3 \)
  - Seller pays: \( 0 - 5 = -5 \)

Mechanism takes $3 from buyer, and gives $5 to the seller!

- Need external subsidy
Problems with VCG

• Difficult to understand
  ➢ Need to reason about what welfare maximizing allocation in agent $i$’s absence

• Does not care about revenue
  ➢ Although we can lower bound its revenue

• With sellers as agents, need subsidy
  ➢ With no subsidy, cannot get the other three properties

• Might be NP-hard to compute
Single-Minded Bidders

• Combinatorial auction for a set of $m$ items $S$

• Each agent $i$ has two private values $(v_i, S_i)$
  - $S_i \subseteq S$ is the set of desired items
  - When given a bundle of items $A_i$, agent has value $v_i$ if $S_i \subseteq A_i$ and 0 otherwise
  - “Single-minded”

• Welfare-maximizing allocation
  - Agent $i$ either gets $S_i$ or nothing
  - Find a subset of players with the highest total value such that their desired sets are disjoint
Single-Minded Bidders

• Weighted Independent Set (WIS) problem
  ➢ Given a graph with weights on nodes, find an independent set of nodes with the maximum weight
  ➢ Known to be NP-hard

• Easy to reduce our problem to WIS
  ➢ Not even $O(m^{0.5-\epsilon})$ approximation of welfare unless $NP \subseteq ZPP$

• Luckily, there’s a simple, $\sqrt{m}$-approximation greedy algorithm
Greedy Algorithm

• **Input:** \((v_i, S_i)\) for each agent \(i\)
• **Output:** Agents with mutually independent \(S_i\)

• **Greedy Algorithm:**
  - Sort the agents in a specific order (we’ll see).
  - Relabel them as 1, 2, …, \(n\) in this order.
  - \(W \leftarrow \emptyset\)
  - For \(i = 1, \ldots, n\):
    - If \(S_i \cap S_j = \emptyset\) for every \(j \in W\), then \(W \leftarrow W \cup \{i\}\)
  - Give agents in \(W\) their desired items.
Greedy Algorithm

• Sort by what?

• We want to satisfy agents with higher values.
  \( v_1 \geq v_2 \geq \cdots \geq v_n \Rightarrow m\text{-approximation} \)

• But we don’t want to exhaust too many items.
  \( \frac{v_1}{|S_1|} \geq \frac{v_2}{|S_2|} \geq \cdots \frac{v_n}{|S_n|} \Rightarrow m\text{-approximation} \)

• \( \sqrt{m}\text{-approximation} : \frac{v_1}{\sqrt{|S_1|}} \geq \frac{v_2}{\sqrt{|S_2|}} \geq \cdots \frac{v_n}{\sqrt{|S_n|}} \) ?

[Lehmann et al. 2011]
Proof of Approximation

• Definitions
  - \( OPT \) = Agents satisfied by the optimal algorithm
  - \( W \) = Agents satisfied by the greedy algorithm
  - For \( i \in W \),
    \[
      OPT_i = \{ j \in OPT, j \geq i : S_i \cap S_j \neq \emptyset \}
    \]

• Claim 1: \( OPT \subseteq \bigcup_{i \in W} OPT_i \)

• Claim 2: It is enough to show that \( \forall i \in W \)
  \[
  \sqrt{m} \cdot v_i \geq \sum_{j \in OPT_i} v_j
  \]

• Observation: For \( j \in OPT_i \), \( v_j \leq v_i \cdot \frac{\sqrt{|S_j|}}{\sqrt{|S_i|}} \)
Proof of Approximation

• Summing over all \( j \in OPT_i \):

\[ \Sigma_{j \in OPT_i} v_j \leq \frac{v_i}{\sqrt{|S_i|}} \cdot \Sigma_{j \in OPT_i} \sqrt{|S_j|} \]

• Using Cauchy-Schwarz (\( \Sigma_i x_i y_i \leq \sqrt{\Sigma_i x_i^2} \cdot \sqrt{\Sigma_i y_i^2} \))

\[ \Sigma_{j \in OPT_i} \sqrt{|S_j|} \cdot 1 \leq \sqrt{|OPT_i|} \cdot \sqrt{\Sigma_{j \in OPT_i} |S_j|} \]
\[ \leq \sqrt{|S_i|} \cdot \sqrt{m} \]
Strategyproofness

- Agent $i$ pays $p_i = v_{j^*} \cdot \sqrt{\frac{|S_i|}{|S_{j^*}|}}$

  - $j^*$ is the smallest index $j$ such that $j$ is currently not selected by greedy but would be selected if we remove $(v_i, S_i)$ from the system

  - **Exercise:** Show that we must have $j^* > i$

  - **Exercise:** Show that $S_i \cap S_{j^*} \neq \emptyset$

  - **Another interpretation:** $p_i = \text{lowest value } i \text{ can report and still win}$
Strategyproofness

• **Critical payment**
  - Charge each agent the lowest value they can report and still win

• **Monotonic allocation**
  - If agent $i$ wins when reporting $(v_i, S_i)$, she must win when reporting $v_i' \geq v_i$ and $S_i' \subseteq S_i$.
  - Greedy allocation rule satisfies this.

• **Theorem:** Critical payment + monotonic allocation rule imply strategyproofness.
Moral

• VCG can sometimes be too difficult to implement
  ➢ May look into approximately maximizing welfare
  ➢ As long as the allocation rule is monotone, we can charge critical payments to achieve strategyproofness
  ➢ Note: approximation is needed for computational reasons

• Later in mechanism design without money...
  ➢ We will not be able to use payments to achieve strategyproofness
  ➢ Hence, we will need to approximate welfare just to get strategyproofness, even without any computational restrictions